Algebra Qualifying Exam
August 23, 2024 (1pm–5pm)

For *page arrangement*, follow these general instructions closely throughout the test.

- Write in a legible fashion.
- For each sheet, write only on a single side and leave a 1×1 square inch area blank at the upper left corner – for staples.
- Start with a new sheet for each question.
- Absolutely no cell phones of any kind during the entire test.
- Arrange all papers by the order of the questions before submission.

The following instructions are given in ″Algebra Qualifying Exam August 2024 Guidelines″ under ″Exam Format″. It is the student's responsibility to read and follow these instructions carefully.

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. No calculators or other electronic devices are allowed.
4. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
5. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.
6. All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.
7. If multiple version of solutions are provided, then all versions will be graded and then the average credits is taken as the final score.
8. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
9. Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.

Exam grade: In order to pass the algebra qualifying exam, you must satisfy the following two criteria:

(a) Receive the full score from at least three questions.

(b) Receive a passing score 70% or better; equivalently, 49 points or more out of the total 70 points.

Notation. Throughout the exam, we use the following notation:

$\mathbb{Z}$ is the ring of integers.
$\mathbb{Q}$ is the field of rational numbers.
$\mathbb{R}$ is the field of real numbers.
$\mathbb{C}$ is the field of complex numbers.
$n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$, where n is a positive integer.
$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n, where n is a positive integer.
$\mathbb{F}_p$ is a field with p elements when p is a prime, i.e. $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$.
S_n is the permutation group on n symbols.
Z_n is a cyclic group order n, usually considered as a multiplicative group.

1. Assume that m and n are relatively prime positive integers.

   (a) (6 pts) Let G be a group of order mn. Assume that G has a unique cyclic subgroup of order m and a unique cyclic subgroup of order n. Prove that G is cyclic and is isomorphic to $Z_m \times Z_n$.

   (b) (4 pts) List all non-isomorphic groups of order 100 by their invariant factor decomposition.

2. Prove through the following steps that, for $2n = 2^a k$ where k is odd, the number of Sylow 2-subgroups of D_{2n} is k.

   Let G be a finite group and p be a prime number dividing the order of $|G|$. Let $P \in \mathrm{Syl_p(G)}$ be a Sylow p-subgroup of G. Let $N_G(P)$ be the normalizer of P in G. A normalizer is a subgroup of G but is not necessarily normal. Write N for $N_G(P)$.

   (a) (3 pts) For any $x, y \in G$, prove that $xPx^{-1} = yPy^{-1}$ if and only if $xN = yN$.

   (b) (2 pts) Let n_p denote the number of distinct Sylow p-subgroups in G for a prime p. Prove that $n_p = \frac{|G|}{|N|}$.

   (c) (4 pts) Let $2n = 2^a k$ where k is odd. Let G be D_{2n} and P a Sylow 2-subgroup. Prove that $N_{D_{2n}}(P) = P$.

   (d) (1 pt) Following (c), prove that $n_2 = k$.

3. (a) (2 pts) Let R denote a commutative ring with 1. Let I denote a principal ideal generated by $a \in R$, i.e. $I = (a)$. Prove or disprove: $I^2 = (a^2)$. Justify your answer.

   (b) (2 pts) In $\mathbb{Z}[x]$, let J denote the ideal generated by 2 and x. Prove that $J^2 \neq (4, x^2)$ by explicitly giving an element in J^2 that is not in the ideal $(4, x^2)$. Justify your answer.

   (c) (6 pts) Let R and S be commutative rings with 1 and let K be an ideal in the direct product ring $R \times S$. Show that $K = L \times M$, where L is an ideal in R and M is an ideal in S.

4. In each of the following rings, determine if the given set is a maximal ideal of the given ring.

   (a) (2 pts) The set $\{2n + 3m : n, m \in \mathbb{Z}\}$ in the ring $\mathbb{Z}$.

   (b) (2 pts) The ideal $(x^5 + 21x^3 + 14)$ in $\mathbb{Q}[x]$.

   (c) (3 pts) The ideal $(x - 2, x^2 + 2)$ in $\mathbb{Z}[x]$.

   (d) (3 pts) The ideal (5) in $\mathbb{Z}[i]$.

5. (a) (4 pts) Work over a commutative ring R with 1. For projective modules, injective modules, and flat modules, state one of the equivalent definitions for each of them, respectively. (You must explain all notation used in the statements, and be specific about the assumptions.)

   (b) (6 pts) Prove that a nontrivial finite abelian group is neither projective, nor injective, nor flat as a module over $\mathbb{Z}$. (You may use the definition of projectivity, injectivity, or flatness that you gave above. If you use a theorem or proposition, you must provide the full statement of the theorem or proposition.)

6. Let V be a n-dimensional vector space ($n \geq 2$) over a field k. Let $T : V \to V$ be a linear transformation. We consider V as a module over $k[x]$ with $xv = T(v)$ for all $v \in V$. Assume that the minimal polynomial of T is $x^3 + x$.

   (a) (3 pts) Determine the annihilator of V in $k[x]$. Recall that the annihilator of a module M in R is defined to be

$$\{r \in R \,|\, rm = 0 \text{ for all } m \text{ in } M\}.$$

   (b) (4 pts) Assume that $n = 5$ and $k = \mathbb{C}$ is the field of complex numbers. Using the Fundamental Theorem of Finitely Generated Modules over a Principal Ideal Domain, list all possible decompositions of V as a direct sum of cyclic modules.

   (c) (3 pt) Repeat Part (b) with the assumption that $k = \mathbb{Q}$, the field of rational numbers, instead.

7. Let $\theta = \sqrt{3 + \sqrt{5}}$.

   (All statements and claims in answering the following questions must be justified.)

   (a) (2 pts) Find the minimal polynomial $f(x)$ of θ over $\mathbb{Q}$.

   (b) (3 pts) Find the splitting field of $f(x)$ over $\mathbb{Q}$.

   (c) (5 pts) Find the structure of the Galois group of $f(x)$ over $\mathbb{Q}$.

8. Let $E = \mathbb{Q}(\sqrt{2}, \gamma)$, where γ satisfies $\gamma^2 - \gamma + 1 = 0$.

   (a) (1 pt) Show that $1 - \gamma$ is the second root of the polynomial $x^2 - x + 1$.

   (b) (3 pts) Prove that $x^2 - x + 1$ is irreducible over $\mathbb{Q}[\sqrt{2}]$. Deduce that $[E : \mathbb{Q}] = 4$. (You must justify all your statements.)

   (c) (3 pts) Let φ be an automorphism of E over $\mathbb{Q}$ defined by $\varphi(\sqrt{2}) = -\sqrt{2}$ and $\varphi(\gamma) = 1-\gamma$. Determine the order of φ and the fixed field of the subgroup generated by φ. Present the fixed field in its final form as simplified as you can get. (Justify your answer.)

   (d) (3 pts) Find the group structure of $\mathrm{Aut(E/\mathbb{Q})}$ and determine whether E is a Galois extension of $\mathbb{Q}$. (Justify your answers.)