Algebra Qualifying Exam

August 21, 1995

There are four pages. A passing grade is 75 out of 100.

Part I (40 points)

Each problem is worth 5 points. Do 8 problems. (If you attempt more than 8 problems, only the first 8 will be graded.)

1 Prove or disprove: If A and B are symmetric $n \times n$ matrices, then $AB = BA$.

2 Let A and B be $n \times n$ matrices such that $AB = I$, the identity matrix. Prove that $BA = I$.

3 Let T be a linear transformation of a finite dimensional vector space over a field F. Let $\lambda \in F$. Recall the definition that λ is an eigenvalue of T iff $\lambda I - T$ is singular. Prove that $\lambda \in F$ is an eigenvalue of T iff for some $v \neq 0 \in V, T(v) = \lambda v$.

4 It is known that the direct product of two groups is a group. Prove or disprove: the direct product of two cyclic groups is a cyclic group.

5 In the group S_n of permutations of an n element set, every permutation may be written uniquely as a product of disjoint cycles (unique up to the order of the cycles). Can every element in $S_n, n \geq 2$ be written as a product of transpositions? *If so* do part (a). *If not*, do part (b).

(a) Prove or disprove that the factorization is unique, i.e. every permutation can be written uniquely as a product of transpositions, except possibly for the order in which the transpositions are written.

(b) Prove the answer is no.

6 A normal series of a group G is a chain (*) of subgroups

$$(*) \qquad G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots \supseteq N_r = \{1\}$$

such that each N_i is normal in N_{i-1}. The factor groups of (*) are the groups N_{i-1}/N_i. The length of the series is the number of strict inclusions.

(a) Find a normal series for Z_6, the cyclic group of order six, of length 2 or greater.

(b) For each of the two or more factor groups in (a), describe a familiar group (not a quotient group) to which the factor group is isomorphic.

(c) Find a normal series for S_6 of length 2 or greater.

(d) For each of the two or more factor groups in (c), describe a familiar group (not a quotient group) to which the factor group is isomorphic.

7 Let P_2 denote the set of all polynomials with real coefficients and degree less than or equal to 2, along with the zero polynomial. In other words

$$P_2 = \{p(x) \in R[x] : deg(p(x)) \leq 2 \text{ or } p(x) = 0\}.$$

(a) Is P_2 a group under $+$?
(b) Is P_2 a group under $\cdot$?
(c) Is P_2 a vector space?
(d) Is P_2 a ring?
(e) Is P_2 an integral domain?
(f) Is P_2 a field?
(g) For any *one* of parts (a) - (f) for which the answer is no, show why. (If you defend the answer to more than one part, only the first one alphabetically which you attempted to answer will be scored.)

8 Prove that D_{12}, the group of symmetries of a regular hexagon, and A_4, the even permutations on 4 symbols, are non-isomorphic groups.

9 Prove that a ring with unity (multiplicative identity) and no left ideals is a division ring.

(Part II begins on the next page.)

Part II (60 points)

Each problem is worth 10 points. Do 6 problems. At least one problem must be chosen from each of the groups A, B and C. (If you do more than 6 problems, the graders will choose the first six which satisfy the condition of one from each group.)

Group A

A 1 Let V be a finite dimensional vector space over a field F.
Give the definition of a linear transformation $T : V \rightarrow V$.
Prove that T is uniquely determined by its values on a basis of V.
Define the rank and nullity of T.
Prove that $rank(T) + nullity(T) = dim(V)$.

A 2 Let $A = (a_{ij})$ be an $n \times n$ matrix over the reals. Give the formal definition of $det(A)$, the determinant of A. Prove from the basic properties of determinants that if A is an $n \times n$ skew symmetrix matrix (i.e. $A^t = -A$), and n is odd then $det(A) = 0$.

A 3 Prove that the eigenvectors belonging to different eigenvalues of a symmetric matrix over the reals are orthogonal.

Group B

B 1 Let $GL(n, R)$ denote the set of all $n \times n$ matrices over R whose determinant is different from zero. Set $SL(n, R) = \{A : A \in GL(n, R) \text{ and } det(A) = 1\}$.
Prove $GL(n, R)$ is a group under matrix multiplication.
Prove that $SL(n, R)$ is a normal subgroup of $GL(n, R)$.
Prove that $GL(n, R)/SL(n, R)$ is isomorphic to the group of nonzero elements of R under multiplication.

B 2 Suppose G is a simple group of order 660. How many elements of order 11 does G contain? Justify your answer.

B 3 Prove that if G is a group of order p^3 where p is a prime, then $G/Z(G)$ is abelian. $Z(G)$, the center of G, is the set of elements of G which commute with all elements of G, i.e.,

$$Z(G) := \{x \in G : xg = gx, \forall g \in G\}.$$

It is known that $Z(G)$ is a normal subgroup of G. You may use any facts about groups of order p or p^2.

Group C

C 1 Define Euclidean domain and show that the Gaussian integers form a Euclidean domain.

C 2 Show $Z[X]$ is a UFD (unique factorization domain) but not a PID (principal ideal domain).

C 3 For this problem $\mathbf{Q}$ is the field of rational numbers.

(a) Prove or disprove: The ring $\mathbf{Q}[x]/ < x^{11} + 1 >$ is a field.

(b) Find the inverse of $x^{10} - x^4 + 1+ < x^{11} + 1 >$ in the ring $\mathbf{Q}[x]/ < x^{11} + 1 >$.