Algebra Qualifying Examination

August 17, 2000

There are three parts to this exam. Part A covers linear algebra, Part B covers group theory, and Part C covers ring theory. In each part you are asked to do a certain number of questions from a larger number of questions. If you do more than the stated number of questions, only the first ones numerically will be scored.

Each problem is worth 10 points. Some of the problems have multiple parts. In these cases, the point values for the different parts are indicated.

The bold-face letters **Z** and **R** refer to the integers and the real numbers, respectively.

Part A **Linear Algebra** Do any five (5) of the eight (8) numbered problems.

1. Let T be a linear transformation with matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

with respect to the standard basis of $\mathbf{R}^3$.

(4) a. Find a basis for the null space of T.

(3) b. Find a basis for the range of T.

(3) c. Is the element $e_1 - e_2 - 2e_3$ in the range of T? (justify your answer)

2. Let T be a linear transformation from an n-dimensional space V to an m-dimensional space W.

(3) a. If $m < n$, can T be one-to-one? Onto? Explain your answers.

(3) b. If $n < m$ can T be one-to-one? Onto? Explain you answers.

(4) c. Prove that if $\{u_1, \ldots, u_n\}$ is a basis for V then $\{Tu_1, \ldots, Tu_n\}$ is a basis for T(V) if and only if T is one-to-one.

3. Suppose that V is a finite dimensional vector space and that $T \in L(V,W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and range $T = \{Tu : u \in U\}$.

4. Suppose $T \in L(V)$ and dim range $T = k$. Prove that T has at most k+1 distinct eigenvalues.

5. Suppose $P \in L(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

6. Suppose V is a complex inner-product space and $T \in L(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

7. Suppose $T \in L(V)$ is invertible. Prove that there exists a polynomial p such that $T^{-1} = p(T)$.

8.

(4) a. Prove or disprove the following statement: If T is a linear transformation on a finite dimensional vector space V and $\lambda_1 \neq \lambda_2$ are distinct eigenvalues with corresponding eigenvectors u_1, u_2, then u_1 and u_2 are necessarily linearly independent.

(3) b. Is this true for more than 2 distinct eigenvalues?

(3) c. What can be said if V is an inner product space and T is normal?

Part B **Group Theory** Do any three (3) out of the first five, **<u>and</u> do problem 6.**

1.

(2) a. Let G be the multiplicative group of positive reals and let H be the additive group of all reals. Prove that log: $G \rightarrow H$ ~~defined by log (x) = e^x~~ is an isomorphism.
↖ base 10 logarithm

(2) b. Let G be a group with identity e, a an element of G and n an integer. If $a^n = e$, prove that the order of a divides n.

(3) c. Prove that any subgroup H of index two in a group G is normal.

(3) d. Let G be the multiplicative group of all n x n non-singular matrices over the reals. Prove that the subset of all matrices with determinant one is a normal subgroup of G.

2.

(3) a. Let G be a group with identity e in which $g^2 = e$ for all g in G. Prove that G is abelian.

(3) b. Prove that the groups **C** and **R** x **R** are isomorphic, where **C** is the complexes and **R** is the reals.

(4) c. If G is a non-abelian group and Z(G) is its center, prove that the factor group G/Z(G) is not cyclic.

3.

(2) a. Let a group G act on a set X (X is also referred to as a G-set). Let x be an element of X. Define the orb(x) and the stabilizer G_x. Prove that G_x is a subgroup of G.

(4) b. Prove the Orbit-Stabilizer Theorem: For a finite group G acting on a set X, the relation $|G|=|orb(x)||G_x|$ holds.

(2) c. Let G be the symmetric group S_n and V be a real vector space with basis $\{v_1, v_2, ..., v_n\}$. For π in G and $v = a_1v_1 + a_2v_2 + .. + a_nv_n$, define

$$\pi.v = a_1v_{\pi(1)} + a_2v_{\pi(2)} + .. + a_nv_{\pi(n)}.$$ Prove that V is a G-set.

(2) d. Find explicitly orb (v) and G_v, when n=4 and $v=v_1+v_2+v_3+v_4$.

4.

(5) a. Prove using the Orbit-Stabilizer Theorem that if p is prime and G is a group of order p^n $(n \geq 1)$, then the center Z(G) of G has more than one element.

(5) b. Let p be a prime. Prove that any group of order p^2 is abelian.

5.

(5) a. If a and b are elements in a group G, the commutator of a and b, is denoted by $[a,b] = aba^{-1}b^{-1}$. The commutator subgroup of G denoted by G' is the subgroup of G generated by all the commutators in G. Prove that H is a normal subgroup of G if and only if G' is contained in H.

(5) b. Prove that the set-theoretic union of two subgroups of a group G is a group if and only if one of them is contained in the other.

6.

(4) a. Let G be a finite group and let p be a prime. Give the definition of a Sylow p-subgroup of G. State the full content of all the Sylow Theorems.

b. What do the Sylow Theorems enable one to deduce about the number of Sylow p-subgroups in the following cases:

(3) i. p=7, $|G| = 28$;
(3) ii p=2, $|G| = 48$.

PART C **Ring Theory** Do any four (4) of the five (5) numbered parts.

1. Let M denote the ring of 2 x 2 matrices over the ring **Z** of integers.
Let $T = \{ \begin{bmatrix} n & 0 \\ 2n & 0 \end{bmatrix} : n \in \mathbf{Z} \}$. Then T is a subring of M. (You need not prove this.)

(4) a. Prove or disprove: T is an ideal of M.

(4) b. Prove that T is isomorphic to the ring **Z**.

(2) c. Find two other subrings of M that are isomorphic to **Z**. (No proofs are necessary)

2. Let I be an ideal of a ring R.

(7) a. If R is a principal ideal ring, prove that R/I is also a principal ideal ring.

(3) b. If R is commutative, we know that R/I is commutative. If R has an identity, then R/I has an identity. Give an example of a ring R with a property *P* (like being a principal ideal ring, or being commutative, or having an identity) and an ideal I such that R/I does **<u>not</u>** have property *P*. (You need only describe R, *P*, and I. No proofs are necessary.)

3. Suppose f is a homomorphism of the ring **Z** onto a field F. What can be said about F? Be as complete as possible.

4. Recall that an ideal P in a ring R is a prime ideal if it has the property that if A and B are ideals of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

(7) a. Suppose P and Q are ideals in a ring R such that $P \not\subseteq Q$ and $Q \not\subseteq P$. Prove that $P \cap Q$ is **<u>NOT</u>** a prime ideal.

(3) b. Give an example of two ideals P and Q in the ring **Z** that meet the conditions of the hypothesis of (a), i.e., $P \not\subseteq Q$ and $Q \not\subseteq P$. (You need only describe P and Q, not prove anything about them.)

5. Let G be an abelian group. Consider the set End G of all endomorphisms of G (homomorphisms of G into G). (End G, +, $\circ$) is a ring where $\circ$ denotes function composition. G is a left End G – module if we define "scalar multiplication" by fa=f(a) for $f \in$ End G and $a \in G$. In arguing that G is a left End G – module, where do we "need" to know that G is abelian? State a property where G being abelian is needed and prove this property.