Algebra Qualifying Exam, Fall 2001

It is possible to obtain a score of 420 on this exam. A **good** score is 300 points. A "good" score guarantee a passing grade.

For the purposes of this exam, an **integral domain** is a commutative ring with unity which has no zero divisors. A **local ring** is a commutative ring with unity which has a unique maximal ideal.

Part I (Short Answer, 10 points each. Do as many as you can. It is possible to score 90; a **good** score is **80** pts.)

Do each of the problems below. Provide short answers or brief computations to defend your work.

1. Consider a matrix,

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ -2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 3 & -1 & 1 & 0 \end{pmatrix}$$

   and then the corresponding linear operator, $T_A(x) = AX$ defined on R^4.

   (a) Find a basis for the null space of T_A.

   (b) What does the result of part (a) say in terms of linear systems of equations.

2. Let $P_2(R)$ denote the real vector space of polynomials with real coefficients and degree at most 2. Let $P_2(R) \to P_2(R)$ be the linear transformation defined b $T(a + bx + cx^2) = a + cx + bx^2$

   (a) Find the matrix T_β corresponding to the ordered basis $\beta = \{1, x, x^2\}$

   (b) Find the eigenvalues and eigenspaces of T.

3. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices need not be invertible if AB is invertible.

4. Let $\alpha = (1, 3, 5, 7, 9, 8, 6)(2, 4, 10)$. What is the smallest positive integer n for which $a^n = a^{-5}$?

5. Let G be the group $< x, y : x^4 = y^2 = (xy)^2 = 1 >$.

   (a) Determine the order of G.

   (b) List the elements in the center of G.

   (c) Find the centralizer of x.

6. Find the multiplicative inverse of 11111 in the ring $Z_{8888\ 80009}$.

7. Construct a field of order 8.

8. Show that $3x^7 + 12x^4 + 8x^2 + 2x + 6$ is irreducible over the integers.

9. Suppose m and n are positive integers which are not perfect squares. Find the inverse of $\sqrt{m} + \sqrt{n}$ in $\mathbf{Q}(\sqrt{m}, \sqrt{n})$. and write it in the form $a\sqrt{m} + b\sqrt{n}$, $a, b \in \mathbb{Q}$.

Part II (Short Proofs, 15 points each. Do as many as you can. It is possible to score 135; a **good** score is **120** pts.)

1. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be linear. Suppose that T is one-to-one and S is a subset of V. Prove that S is linear independent iff $T(S)$ is linearly independent.

2. Prove that a linear operator on a finite dimensional vector space is one-to-one if and only if it is onto.

3. $T : V \to W$ be a module homomorphism.

   (a) Define: *null space* of T. (An alternate word for "null space" is "kernel".)

   (b) Prove the null space is a submodule of V.

   (c) Prove that if S is a subspace of W then $T^{-1}(S)$ is a subspace of V.

4. Suppose G is non abelian of order p^3 where p is prime. Prove that if $Z(G) \neq e$ then $|Z(G)| = p$. (Here $Z(G)$ represents the center of G.)

5. Prove that a group of order 105 contains a subgroup of order 35.

6. Prove that if G is cyclic, then every subgroup of G is cyclic.

7. Prove that is M is an ideal of a ring R and $m \in M$ is a unit, then $M = R$.

8. Prove that in a PID (principal ideal domain), every prime ideal is maximal.

9. Prove that a finite integral domain is a field

Part III (Proofs, 20 points each. Do as many as you can. It is possible to score 200; a **good** score is **100** pts.)

1. Prove that any matrix of rank one, with m rows and n columns, can be written as a matrix product, $v \cdot w$, where v is an $m \times 1$ matrix and w is an $1 \times n$ matrix. (All matrices may assumed to be over an arbitrary field, F.)

2. Let $L : V \to V$ be a linear transformation of a vector space V over a field of characteristic 0. Suppose that $T^2 = 2T$. Show that V is a direct sum of the null space and the range space of T. (A vector space V is a direct sum of subspaces W and U if $W \cap U = \emptyset$ and $W + U = V$.)

3. A $n \times n$ matrix, A, is *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$.

   (a) Show that set, S, of and the set, T, of all skew-symmetric $n \times n$ matrices are subspaces of the vector space of all $n \times n$ matrices.

   (b) Show that every $n \times n$ matrix can be represented as a sum of a symmetric matrix and a skew-symmetric matrix.

   (c) Show that, if the characteristic of the underlying field is zero, then the vector space of all $n \times n$ matrices is a direct sum of S and T.

4. (a) Let G be the multiplicative group of units modulo 24, that is, $G = U(24) := \{1, 5, 7, 11, 13, 17, 19, 23\}$. Let $H = \{1, 17\}, K = \{1, 13\}$. Compute the set HK. Is HK a group?

   (b) Let G $= S_3$, the symmetric group on the set $\{1, 2, 3\}$. Let $H = \{e, (12)\}$ and $K = \{e, (13)\}$. Compute the set HK. Is HK a group?

   (c) Suppose H and K are subgroups of a group G. Find necessary and sufficient conditions such that HK is a subgroup of G.

5. Prove that if K and L are submodules of an R-module M, with $K \leq L$, then M/L is isomorphic to $(M/K)/(L/K)$.

6. Prove that

   (a) every field has characteristic zero or p (where p is a prime.)

   (b) every field E may be viewed as a vector space over a field F where F is either isomorphic to the ring Z_p (p a prime), or F is isomorphic to the rational numbers $\mathbf{Q}$.

7. (a) Find the unique member of $Z/(521 \cdot 641)\mathbf{Z}$ which is a subset of

$$(207 + 521\mathbf{Z}) \cap (128 + 641\mathbf{Z}).$$

(b) Find the unique member of $\mathbf{Q}[x]/((x^2+1)(2x)\mathbf{Q}[x])$ which is a subset of

$$(x + (x^2 + 1)\mathbf{Q}[x]) \cap (7 + (2x)\mathbf{Q}[x]).$$

(c) Prove that if R is a commutative ring R with unity, and A and B are ideals of R, such that $A + B = R$ then, given cosets $x + A$ and $y + B$, there is a unique member of $R/(A \cap B)$ that is contained in both $x + A$ and $y + B$.

8. A Boolean ring R is a ring with the property that $a^2 = a$ for all $a \in R$. Prove

(a) that every Boolean ring has characteristic 2,

(b) that every Boolean ring is commutative,

(c) that every finite Boolean ring is isomorphic to the collection of subsets of a set under the operations of symmetric difference and intersection.

9. (a) Give an example of a finite local ring which is not a field.

(b) Give an example of an infinite local ring which is not a field.

(c) Prove that in a local ring R with maximal ideal M, the set of units is precisely those elements of R which are not in M.

(d) Given an example of a ring with unity, which has a maximal ideal M, yet has the property that there exist nonunits not in M.

(e) Suppose R is a commutative ring with unity, and R has the property that the set of nonunits form an ideal. Prove R is a local ring.

10. For each field E below, compute the minimal polynomial over the rationals, $\mathbf{Q}$, find the Galois group of the minimal polynomial, and find all subfields of the field E. Then give the Galois correspondence between the subfields of E and the subgroups of $Gal_{\mathbf{Q}}E$.

(a) $\mathbf{Q}(e^{\pi i/6})$.

(b) $\mathbf{Q}(\sqrt{p}, \sqrt{q})$, where p and q are distinct integer primes.