August 2004
Algebra Qualifying Exam

There are 90 points possible on this exam, a passing grade is typically 70 percent (63 points). Partial credit will be given sparingly - getting entire problems correct is better than amassing partial credit. Be aware that points will be taken for wrong statements. If two answers are given and one of them is wrong, no credit will be given.

No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

MTH 525 - Linear Algebra

Do three out of the following four problems.

1. Let V be the vector space of real polynomials in one variable of degree less than or equal to 2. Let $\mathscr{B}_1 = \{1, x, x^2\}$ be the standard basis for V.
   (5) (a) Show that $\mathscr{B}_2 = \{3x^2+3, x+2, x^2+x+1\}$ is a basis for V.
   (5) (b) Find a matrix P so that for any $v \in V$ we have $P[v]_{\mathscr{B}_1} = [v]_{\mathscr{B}_2}$.

2. Let V and W be finite dimensional vector spaces over a field F.
   (5) (a) Given linear transformations $T \in L(V,W)$ and $S \in L(W,V)$, show that $\text{rank}(S \circ T) \leq \text{rank}(T)$.
   (5) (b) Suppose that $T : W \to V$ is one-to-one. Prove that the transpose $T^t : V^* \to W^*$ is onto.

3. Let A be an $m \times m$ matrix, and suppose that $A^n = 0$ for some $n \in \mathbb{N}$.
   (3) (a) Show that the only eigenvalue of A is 0.
   (4) (b) Show that $A^m = 0$.
   (3) (c) If $m = 3$, give all possible Jordan forms of A.

4. Let V be an inner product space over $\mathbb{C}$ (note that $\dim(V)$ is not assumed to be finite) and let $T : V \to V$ be a linear operator. We denote the *adjoint* of T by T^*.
   (5) (a) Show that if $TT^* = T^*T$, then the range of T is orthogonal to the nullspace of T.
   (5) (b) Let T be a self-adjoint operator on V (i.e. $T = T^*$). Show that if $T(v) = cv$, then $c\langle v, v\rangle = \bar{c}\langle v, v\rangle$. Conclude that characteristic values of T are real.

MTH 623 - Group Theory

Do two out of the following four problems.

(12) 5. Prove that every group of order 176 is solvable. You may **NOT** use Burnside's Theorem that says that any group of order $p^r q^s$ is solvable.

(12) 6. Let G be a group with a normal subgroup N of order 5, such that $G/N \cong S_3$. Show that G has a normal subgroup of order 15, and that G has 3 subgroups of order 10 that are not normal.

(12) 7. Let G be a finite group and let K be a subgroup of index p, where p is the smallest prime dividing the order of G. Show that K is normal in G. [Hint: Show that there is a group homomorphism from G to S_p, the symmetric group on p elements].

8. Let G be an abelian group with the following property:

(*) If $H < G$ then there exists a subgroup K of G such that $G = H \oplus K$

Show each of the following:

(2) (a) Each subgroup of G also has property (*).

(5) (b) Each element of G has finite order.

(5) (c) If p is a prime then G does not have an element of order p^2.

MTH 625 - Ring Theory

All rings are assumed to be with identity.

Do two out of the following four problems.

(12) 9. Let R be a ring, and let P be a left R-module. Use the following definition of a projective module: P is *projective* if for each epimorphism of left R-modules $f : M \to M'$ and for any homomorphism of left R-modules $g : P \to M'$, there exits a homomorphism of left R-modules $h : P \to M$ such that $f \circ h = g$. Prove: P is projective if and only if every exact sequence of left R-modules

$$0 \to A \to B \to P \to 0$$

splits.

(12) 10. The Jacobson radical $J(R)$ of a ring R is defined to be the intersection of all maximal ideals of R. Let R be a commutative ring. Prove that $x \in J(R)$ if and only if $1 - xy$ is a unit for all $y \in R$.

(12) 11. Let R be a commutative ring such that for every $x \in R$ there is an integer $n > 1$ (depending on x) such that $x^n = x$. Show that every prime ideal in R is maximal.

(12) 12. Show that if p is a prime such that $p \equiv 1 \bmod 4$, then $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.

Field Theory

Do either problem 13 or problems 14 and 15.

13. Let K be a field of prime characteristic p. Let $F = K(t)$ be the field of rational functions on the indeterminate t. Let $f(x) = x^{2p} - tx^p + t \in F[x]$. Note that $f(x)$ is irreducible in $F[x]$.

(6) (a) Let $E = F[s]$, where s is a root of the polynomial $x^p - t \in F[x]$. Let L be a splitting field for $f(x)$ over E, prove that $[L : E] \leq 2$.

(6) (b) Show that $L = F[\alpha]$, where α is a root of $f(x)$. [Hint: Use the relationships between the coefficients of a polynomial of degree 2 and its roots].

(6) 14. Define what it means for a field to be *perfect* and show that every field F of characteristic 0 is perfect.

(6) 15. Let $\mathbb{Q}$ be the field of rational numbers, and let ϵ be a complex primitive 9^{th} root of unity. Determine the Galois group, G, of $\mathbb{Q}(\epsilon)$ over $\mathbb{Q}$ and all the intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\epsilon)$.