August 2005
Algebra Qualifying Exam

There are 100 points possible on this exam, a passing grade is typically 70 percent (70 points). Partial credit will be given sparingly - getting entire problems correct is better than amassing partial credit. Be aware that points will be taken for wrong statements. If two answers are given and one of them is wrong, no credit will be given.

No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

Notation: Note that throughout the exam $\mathbb{Z}_n$ is used to denote $\mathbb{Z}/n\mathbb{Z}$.

MTH 525 - Linear Algebra

Do three out of the following four problems.

(10) 1. Let A be an $n \times n$ matrix and suppose that A is not invertible. Show that there exists a non-zero matrix B such that $AB = BA = 0$.

(10) 2. Let A be an $n \times n$ matrix with complex entries. Prove that if every characteristic value of A is real, then A is similar to a matrix with real entries.

3. Let A be an $n \times n$ matrix.

(3) (a) Show that A is diagonalizable if and only if $A - I$ is diagonalizable.

(3) (b) If $p(x)$ is the characteristic polynomial of A, show that $p(x+1)$ is the characteristic polynomial of $A - I$.

(4) (c) Suppose A is a 2×2 matrix with $\det(A) = 0$. If $\det(A - I) = 0$, what is $\operatorname{tr}(A)$?

4. Let V and W be finite dimensional vector spaces over a field F.

(5) (a) Given linear transformations $T \in L(V, W)$ and $S \in L(W, V)$, show that $\operatorname{rank}(S \circ T) \leq \operatorname{rank}(S)$.

(5) (b) Suppose that $T : W \to V$ is onto. Prove that the transpose $T^t : V^* \to W^*$ is one-to-one.

MTH 623 - Group Theory

Do three out of the following five problems.

5. We examine the alternating group $G = A_5$.

(3) (a) For each conjugacy class C of A_5, give a representative element $\alpha \in C$ and give the size of the class C.

(3) (b) For each conjugacy class C of A_5 and representative element $\alpha \in C$, give the cardinality of the centralizer $C_G(\alpha)$.

(4) (c) Argue from your work above that A_5 is simple.

6. Let G be a nonabelian group of order 105.

(1) (a) How many Sylow-5 subgroups could G have? How many Sylow-7 subgroups?

(1) (b) How many Sylow-3 subgroups could G have?

(2) (c) Suppose a Sylow-3 subgroup $P =< \alpha >$ is not normal. Compute the size of $N(P)$, the normalizer of P in G.

(2) (d) Suppose α_1 is an element of order 3 and α_2 is a conjugate of α_1. Let $P_1 =< \alpha_1 >$ and $P_2 =< \alpha_2 >$ and assume P_1 is not normal. Show that $N(P_1) \cap N(P_2)$ has order 5.

(2) (e) Let $\alpha \in G$ be any element of order three and $\beta \in G$ any element of order five. Show that α and β commute.

(2) (f) Show that there is just one nonabelian group of order 105. Describe it.

7. Let $GL(n, K)$ represent the set of nonsingular $n \times n$ matrices over the field K and $SL(n, K)$ be the subset of $GL(n, K)$ consisting of the matrices with determinant one.

(5) (a) Prove that $SL(n, K)$ is a normal subgroup of $GL(n, K)$ and that $GL(n, K)/SL(n, K)$ is isomorphic to the group $(K^*, \cdot)$ of nonzero field elements under multiplication.

(5) (b) Let K be the field of order nine and set $n = 2$. Describe the composition series of $GL(n, K)$. Explicitly describe each factor of the composition series, giving its size and isomorphism class.

8. Let G be a group and G' the derived subgroup of G. (Recall that G' is generated by the commutators $[a, b] = aba^{-1}b^{-1}$ of G.)

(5) (a) Prove that for any normal subgroup K of G, G/K is abelian if and only if $K \leq G'$.

(5) (b) Show that if G' is solvable then so is G.

9. Set $A := \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Note that A has order three under multiplication. Given a field K of order p, let V be a vector space of dimension two over K. We may view A as representing a nonsingular linear transformation from V onto V.

(5) (a) Use this idea to explicitly describe a semidirect product of $\mathbb{Z}_p \times \mathbb{Z}_p$ by $\mathbb{Z}_3$. (Here p is a prime integer.)

(5) (b) If, above, we set $p = 2$, we obtain a semidirect product of the Klein-4 group by $\mathbb{Z}_3$. This group should be recognized as a familiar group of order 12. Which one?

MTH 625 - Ring Theory

All rings are assumed to have an identity.

Do either problem 10 or problem 11, and do one of problems 12, 13, or 14.

10. Let $f : R \to S$ be a ring epimorphism with kernel K. [The rings R and S are not assumed to be commutative]. Prove:

(3) (a) If P is a prime ideal in R that contains K, then $f(P)$ is a prime ideal in S.

(3) (b) There is a one-to-one correspondence between the set of prime ideals in R that contain K and the set of all prime ideals in S.

(4) (c) If I is an ideal in R then every prime ideal in R/I is of the form P/I where P is a prime ideal in R that contains I.

(10) 11. Let R be an integral domain and for each maximal ideal M, consider the subring R_M of the quotient field of R. Show that $R = \bigcap_M R_M$ where M runs over all maximal ideals of R.

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(10) 12. Let R be a ring. Prove that if $f : A \to B$ and $g : B \to A$ are R-module homomorphisms such that $gf = 1_A$, then $B = \text{Image}(f) \oplus \ker(g)$.

(10) 13. Prove that the set $\mathbb{Q}$ of rational numbers is not a free $\mathbb{Z}$-module.

14. For any abelian group A and any positive integers m, n show:

(5) (a) $\text{Hom}(\mathbb{Z}_m, A) = A[m] := \{a \in A | ma = 0\}$.

(5) (b) $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{(m,n)}$ where (m, n) is the greatest common divisor of m and n.

Field Theory

Do problem 15, and do either problem 16 or problem 17.

15. Let F be an algebraic closure of $\mathbb{Q}$, the field of rational numbers, and let $E \subset F$ be a splitting field for the set of polynomials $S = \{x^2 + a | a \in \mathbb{Q}\}$. Prove the following:

(3) (a) $E = \mathbb{Q}(X)$ where $X = \{\sqrt{p} | p = -1 \text{ or } p \text{ is a prime integer}\}$.

(3) (b) If $\sigma \in \text{Aut}_{\mathbb{Q}} E$, then $\sigma^2 = 1_E$, and therefore $\text{Aut}_{\mathbb{Q}} E$ is a vector space over $\mathbb{Z}_2$.

(4) (c) $\text{Aut}_{\mathbb{Q}} E$ is not denumerable (i.e. uncountable). Hint: Show that the cardinality of $\text{Aut}_{\mathbb{Q}} E$ is the cardinality of the power set of X.

(10) 16. Let K be a field and let f a separable cubic polynomial in $K[x]$ with Galois group S_3 and roots u_1, u_2 and u_3 in some algebraic closure of K. Let F be a splitting field of f. Determine the distinct intermediate field extensions of K by F, and the corresponding subgroups of the Galois group.

(10) 17. Determine all normal subgroups of the Galois group and the corresponding subfields of the splitting field over $\mathbb{Q}$ of $f(x) = (x^3 - 2)(x^2 - 3)$.