Algebra Qualifying Exam
August 29, 2008

Do all six problems. The exam is 50 points. A passing grade is 35/50. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. (9 points)
   Let $k \geq 1$ be an integer, and let p be an odd prime. **Prove**: No group of order $8p^k$ is simple. (**Note:** If you use Burnside's Theorem, you should prove it. Alternately, if you use the Classification of Finite Simple Groups, you should prove it.)

2. (6 points)
   Suppose that G is a group of order 12 and that P is a 3-Sylow subgroup of G.

   (a) **Prove**: If P is normal in G and $P = \langle x \rangle$, then $|C_G(x)| \geq 6$.

   (b) **Prove**: If P is normal in G, then $Z(G)$ has an element of order 2.

   (c) **Prove**: If $Z(G)$ has no element of order 2, then $G \cong A_4$.

   Recall that a ring R satisfies the maximal condition on ideals if every non-empty subset of ideals of R contains a maximal element with respect to inclusion.

3. (10 points)
   Let R be an integral domain and write (a) for the principal ideal generated by $a \in R$. Recall that an element of R is said to be irreducible if it is nonzero, not a unit, and has no proper factorization.

   i. Show that $(a) \subseteq (b)$ if and only if $b|a$, and that $(a) = (b)$ if and only if $b = au$ for some unit $u \in R$. (2 points)

   ii. If R is a UFD (unique factorization domain), prove that the set of principal ideals of R satisfies the maximal condition. (4 points)

   iii. If the set of principal ideals of R satisfies the maximal condition, show that every nonzero, nonunit element of R can be written as a finite product of irreducible elements. (4 points)

Continued on the other side.

4. (5 points) Let F be a field and let $f(x)$ be an irreducible polynomial in $F[x]$. Show that if K is a Galois extension of F, then all irreducible factors of $f(x)$ in $K[x]$ have the same degree.

5. (5 points) Find the Galois group of $f(x) = x^4 - x^2 + 1$ over $\mathbb{Q}$.

6. (15 points) Find the characteristic polynomial, the minimal polynomial, and the Jordan canonical form for the real matrix A, given below. Find a basis of $\mathbb{R}^3$ relative to which A is in its Jordan canonical form.

$$A = \begin{bmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{bmatrix}.$$