Algebra Qualifying Exam
August, 2009

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Prove that $(\mathbb{Z}/32\mathbb{Z})^\times$ is not cyclic.

2. Let D_{16} be the dihedral group of order 16 and let s and r be the commonly used generators for reflection and rotation respectively. Let H denote the subgroup generated by r^4 and sr^2.

   (a) Determine the isomorphism type of H.

   (b) It is known that D_{16} has three subgroups of order 8:

   $$\langle s, r^2 \rangle, \ \langle r \rangle, \ \langle sr, r^2 \rangle$$

   Determine the centralizer $C_{D_{16}}(H)$ and the normalizer $N_{D_{16}}(H)$.

3. (a) Prove that no group of order $84 = 2^2 \cdot 3 \cdot 7$ is simple.

   (b) Let G be a group of order $2^k \cdot 3 \cdot 7$. Follow steps i through iii to prove that if $k \geq 19$, then no group of order $2^k \cdot 3 \cdot 7$ is simple.

   We use $\mathcal{P}$ to denote the set of all Sylow 2-subgroups of G.

   i. Describe the number of elements in $\mathcal{P}$.

   ii. Prove that there exists a group homomorphism $\varphi : G \longrightarrow S_{21}$ induced by conjugating elements in $\mathcal{P}$.

   iii. It is a fact that $2^{19} \nmid 21!$. (You may assume this without proof.) Prove that if $k \geq 19$, then the group homomorphism φ is not injective. Deduce that G is not a simple group.

4. Let $x^3 - 2x + 1$ be an element of the polynomial ring $E = \mathbb{Z}[x]$ and use the bar notation to denote passage to the quotient ring $\mathbb{Z}[x]/(x^3 - 2x + 1)$. Let $p(x) = x^3 + 2x^2 - 1$ and let $q(x) = (x - 1)^4$.

   (a) Express each of $\overline{p(x) + q(x)}$ and $\overline{p(x)q(x)}$ in the form of $\overline{f(x)}$ for some polynomial $f(x)$ of degree ≤ 2.

   (b) Prove that $\overline{E}$ is not an integral domain.

   (c) Prove that $\overline{x}$ is a unit in $\overline{E}$.

5. Prove that a finite integral domain is a field. Deduce that if R is a finite commutative ring with identity, then every prime ideal of R is a maximal ideal.

6. (a) Determine the splitting field of $x^3 - 2$ over $\mathbb{Q}$, denoted E.

   (b) Let G be the Galois group of E over $\mathbb{Q}$. For each subgroup of G, including G itself, determine its corresponding fixed field.

7. (a) Find the minimal polynomial of $\sqrt{5 + 2\sqrt{6}}$ over $\mathbb{Q}$. Determine the degree of the extension field $\mathbb{Q}(\sqrt{5 + 2\sqrt{6}})$ over $\mathbb{Q}$. (You must address the irreducibility of the polynomial.)

   (b) Prove that the extension in part (a) is a Galois extension, and determine the isomorphism type of its Galois group.