Algebra Qualifying Exam
August 2010

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. As usual, let S_n and A_n denote the symmetric group on n letters and the alternating group on n letters respectively. Z_n denotes the cyclic group of order n.
   (a) How many elements are in the conjugacy class of (123) in S_4? How many elements are in the conjugacy class of (123) in A_4?
   (b) Prove that A_4 has no subgroups of order 6.
   (c) Are the groups A_4 and $Z_2 \times S_3$ isomorphic? Explain why or why not.
2. Prove the following properties of a Principal Ideal Domain.
   (a) Prove that any two nonzero elements of a P.I.D. have a least common multiple.
   (b) Prove that a quotient of a P.I.D. by a prime ideal is again a P.I.D.
3. (a) Prove that if the Galois group of the splitting field of a cubic over $\mathbb{Q}$ is the cyclic group of order 3, then all the roots of the cubic are real.
   (b) By an example, show that the Galois group of the splitting field of a cubic over $\mathbb{Q}$ can have order greater than 3.
   (c) By an example, show that the Galois group of the splitting field of a cubic over $\mathbb{Q}$ can have order less than 3.
4. Assume that k is a positive integer and G is a group of order $3^k \cdot 7$.
   (a) **Prove:** If $k \leq 5$, then G has a normal subgroup of order 7.
   (b) **Prove:** If $k \geq 6$, then G has a normal subgroup H such that the order of H is divisible by 3^{k-2}.

5. Let I be an ideal of the commutative ring R. Define the *radical* of I to be

$$\operatorname{rad} I := \{r \in R | r^m \in I \text{ for some } m \in \mathbb{Z}^+\}.$$

(a) Prove that $\operatorname{rad} I$ is an ideal of R.

(b) An ideal I of R is called a *radical ideal* if $\operatorname{rad} I = I$. Prove that every prime ideal of R is a radical ideal.

(c) Let $n > 1$ be an integer. Prove that 0 is a radical ideal in $\mathbb{Z}/n\mathbb{Z}$ if and only if n is a product of distinct primes to the first power (*i.e.* n is square free). Deduce that (n) is a radical ideal of $\mathbb{Z}$ if and only if n is a product of distinct primes in $\mathbb{Z}$.

6. (a) Prove that $\mathbb{Q}[\sqrt{2}+\sqrt{5}] = \mathbb{Q}[\sqrt{2}, \sqrt{5}]$. Conclude that $[\mathbb{Q}(\sqrt{2}+\sqrt{5}) : \mathbb{Q}] = 4$.

(b) Find the minimal polynomial of $\sqrt{2}+\sqrt{5}$. You must verify that the polynomial is irreducible.

(c) Is $\mathbb{Q}[\sqrt{2}+\sqrt{5}]$ the splitting field of the polynomial you just found? Justify your answer if yes, or else find its splitting field.

7. Let G be a group and let $\operatorname{Aut}(G)$ denote the group of all automorphisms of G. A subgroup H of G is *characteristic* if $\sigma(H) = H$ for all $\sigma \in \operatorname{Aut}(G)$.

(a) Prove: If $G = HK$, where H and K are characteristic subgroups of G, and if $H \cap K = 1$, then $\operatorname{Aut}(G) \cong \operatorname{Aut}(H) \times \operatorname{Aut}(K)$.

(b) Describe the automorphism group of $Z_5 \times Z_7$. (Z_n denotes the cyclic group of order n.)