Algebra Qualifying Exam
August 26, 2011

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

Notation: D_{2n} is the dihedral group of order $2n$. S_n is the symmetric group on n elements. $n_p(G)$ is the number of Sylow p-subgroups of G. $\mathbb{F}_q$ is a field consisting of q elements.

1. (10 points)

   (a) Assume the following presentation for the dihedral group of order 8:

   $$D_8 = \langle s, t \mid s^2 = t^4 = 1,\ sts^{-1} = t^3 \rangle.$$

   Prove there exists a subgroup of S_4 that is isomorphic to D_8.

   (b) By Part (a), we may identify D_8 as a subgroup of S_4. Prove that the only element in D_8 that commutes with (123) is the identity element.

2. (10 points) For each of the following rings, determine if the designated set is an ideal of the given ring. Justify your answers.

   (a) The set $\{(a, -a) | a \in \mathbb{Z}\}$ in the ring $\mathbb{Z} \times \mathbb{Z}$.

   (b) The set of all polynomials whose constant term is a multiple of 3 in the ring $\mathbb{Z}[x]$.

   (c) The set of all polynomials whose coefficient of x^2 is a multiple of 3 in the ring $\mathbb{Z}[x]$.

   (d) The set of all nilpotent elements in $M_2(\mathbb{Z})$, where $M_2(\mathbb{Z})$ is the ring of all 2×2 matrices with entries in $\mathbb{Z}$. Recall that an element x is nilpotent if $x^n = 0$ for some positive integer n.

3. (10 points)

   (a) Find the minimal polynomial $m(x)$ of $\sqrt{2+\sqrt{2}}$ over $\mathbb{Q}$. (You must justify that the polynomial you find is irreducible to claim it is minimal.)

   (b) Prove that the splitting field of $m(x)$ over $\mathbb{Q}$ is a cyclic extension; namely the Galois group of the extension field is cyclic.

(c) Draw the lattice of subgroups of the Galois group of $\mathbb{Q}(m(x))$ and the lattice of their corresponding fixed fields.

4. (10 points) Let $k \geq 1$ be an integer. **Prove**: No group of order $2 \cdot 3 \cdot 5^k$ is simple. (**Note:** If you use the Classification of Finite Simple Groups, you should prove it.)

5. (10 points) Let $R = \mathbb{Z}[\sqrt{5}] = \{a+b\sqrt{-5} : a, b \in \mathbb{Z}\}$. Recall that $N(a+b\sqrt{-5}) = a^2+5b^2$ is a norm on R. Consider the ideal $I = (2, 1+\sqrt{-5})$ in R.

   (a) Prove that I is not a principal ideal. **Hint**: Suppose that $I = (\alpha)$. Prove that $N(\alpha) = 2$, and use this to get a contradiction.

   (b) Prove that $I^2 = (2)$, so I^2 is a principal ideal.

6. (10 points)

   (a) Is $\mathbb{F}_5[x]/(x^3 + 2x + 1)$ a field? Explain why or why not.

   (b) How many elements are in $\mathbb{F}_5[x]/(x^3 + 2x + 1)$?

   (c) Construct a finite field with 27 elements.

7. (10 points) Suppose that G is a simple group of order $90 = 2 \cdot 3^2 \cdot 5$.

   (a) Prove that $n_5(G) = 6$. Deduce that if P is a Sylow 5-subgroup of G, then $|N_G(P)| = 15$, and G is isomorphic to a subgroup of S_6.

   (b) Explain why S_6 has 144 elements of order 5. Deduce that $n_5(S_6) = 36$. Use this to prove that if Q is a Sylow 5-subgroup of S_6 then $|N_{S_6}(Q)| = 20$.

   (c) Explain why parts (a) and (b) contradict each other, and deduce that there are no simple groups of order 90.