Algebra Qualifying Exam
August 24, 2012

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Prove that if G is a group and $|G| = 2^2 \cdot 3 \cdot 11^k$ for some integer $k \geq 1$, then G is not simple. (If you use the classification of finite simple groups, you should prove it.)

2. An element z of a ring R is said to be nilpotent if $z^m = 0$ for some $m \in \mathbb{Z}_+$.

   (a) Show that 0 is the only nilpotent in an integral domain.

   (b) Show that if $n = a^k b$ for some integers a and b then $\overline{ab}$ is a nilpotent element of $\mathbb{Z}/n\mathbb{Z}$.

   (c) If $a \in \mathbb{Z}$ is an integer, show that the element $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent if and only if every prime divisor of n is also a divisor of a. Using this or otherwise, determine the nilpotent elements of $\mathbb{Z}/72\mathbb{Z}$ explicitly.

3. Find the Galois group of $x^4 - 12x^2 + 18$ over $\mathbb{Q}$. Give the size of the group, and express the group in terms of generators and relations.

4. A group G is said to be *solvable* if there are subgroups

$$\{1\} = G_n \subset G_{n-1} \subset \cdots \subset G_1 \subset G_0 = G$$

   such that
   (i) G_i is normal in G_{i-1}, and
   (ii) $|G_{i-1}/G_i|$ is prime.

   (a) Show that $\{(12)(34), (13)(24), (14)(23)\}$ generates a normal subgroup in S_4.

   (b) Show that A_4 and S_4 are solvable.

5. Let R be a commutative ring with unity. For an ideal I of R, define

$$\text{Rad}(I) = \{x \in R : x^m \in I \text{ for some positive integer } m\}$$

(a) Let I and J be ideals of R. Prove that $IJ \subseteq I \cap J$. Is it necessarily true that $I \cap J \subseteq IJ$? Prove it or give a counter-example.

(b) **Prove**: $\text{Rad}(IJ) = \text{Rad}(I \cap J) = \text{Rad}(I) \cap \text{Rad}(J)$.

(c) **Prove**: If P is a prime ideal and $I \subseteq P$, then $\text{Rad}(I) \subseteq P$.

6. Consider the extension $\mathbb{Q} \subset L = \mathbb{Q}(\sqrt[4]{2}, \sqrt[3]{3})$. We will compute $[L : \mathbb{Q}]$.

(a) Show that $x^4 - 2$ and $x^3 - 3$ are irreducible over $\mathbb{Q}$.

(b) Use $\mathbb{Q} \subset \mathbb{Q}(\sqrt[4]{2}) \subset L$ to show that $4|[L : \mathbb{Q}]$ and $[L : \mathbb{Q}] \leq 12$.

(c) Use $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{3}) \subset L$ to show that $[L : \mathbb{Q}]$ is also divisible by 3.

(d) Explain why parts (b) and (c) imply that $[L : \mathbb{Q}] = 12$.

7. Let G be a group. The *commutator subgroup* of G is the group

$$G' = \langle xyx^{-1}y^{-1} : x, y \in G \rangle.$$

(a) Recall that a subgroup K of G is *characteristic* if for every automorphism $\phi : G \to G$, $\phi(K) \subseteq K$. **Prove**: G' is a characteristic subgroup of G.

(b) **Prove**: If N is a normal subgroup of G and G/N is abelian, then G' is a subgroup of N.

(c) **Prove**: If H is a subgroup of G and $G' \subseteq H$, then H is normal in G.