Algebra Qualifying Exam
August 19, 2013

Do all seven problems. Each problem is worth 10 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let G be a group of order $435 = 3 \cdot 5 \cdot 29$. You may assume the following theorem:

   Suppose H is a group with subgroups $L_1, \dots, L_s$ such that $L_1, \dots, L_s$ are all normal in H and $L_1 \cdots L_k \cap L_{k+1} = 1$ for $k = 1, \dots, s-1$. Then $L_1 L_2 \cdots L_s \cong L_1 \times L_2 \times \cdots \times L_s$.

   (a) Prove that G has a normal subgroup P of order 5 and a normal subgroup R of order 29.

   (b) Let Q be a Sylow 3-subgroup of G. Prove that PQ is a cyclic subgroup of G of order 15.

   (c) Prove that G has a normal subgroup of order 3.

   (d) Prove that G is cyclic.

2. Determine the degree of the extension $\mathbb{Q}\left(\sqrt{\frac{1+\sqrt{-3}}{2}}\right)$ over $\mathbb{Q}$. Is this extension a Galois extension? If so, what is its Galois group?

3. (a) Prove that in a Principal Ideal Domain (PID) a nonzero element is irreducible if and only if it is a prime.

   (b) Consider the ring $\mathbb{Z}[\sqrt{-3}]$ with the usual norm $N(a + b\sqrt{-3}) = a^2 + 3b^2$. Show that $\mathbb{Z}[\sqrt{-3}]$ is not a PID by showing that $1 + \sqrt{-3}$ is irreducible but not prime.

4. A group G is said to be *metabelian* if there exists a normal subgroup N of G such that N and G/N are both Abelian. Prove that any group of order 12 is metabelian.

5. (a) Let $f(x)$ be a polynomial in $F[x]$, where F is a field. Prove that $F[x]/((f(x))$ is a field if and only if $f(x)$ is irreducible.
   (b) Prove that $\mathbb{F}_5[x]/(x^3 + 3x + 3)$ is a field.
   (c) Determine the number of elements in the field $\mathbb{F}_5[x]/(x^3 + 3x + 3)$.

6. Let $f : A \to B$ be a ring homomorphism. Recall that if $X \subseteq A$ and $Y \subseteq B$, then

$$f(X) = \{f(x) : x \in X\} \text{ and } f^{-1}(Y) = \{a \in A : f(a) \in Y\}.$$

Let I be an ideal of A and J be an ideal of B. We define the *extension* I^e of I to be the ideal of B generated by $f(I)$: that is,

$$I^e := \left\{ \sum_{i=1}^{n} y_i f(x_i) : x_1, \ldots, x_n \in I, y_1, \ldots, y_n \in B \right\}.$$

Similarly, we define the *contraction* J^c of J to be $f^{-1}(J)$.

   (a) Prove that J^c is always an ideal of A.
   (b) Give an example to show that $f(I)$ is not necessarily an ideal of B.
   (c) Prove that if J is prime, then either J^c is a prime ideal of A or $J^c = A$.
   (d) Give an example to show that if I is a prime ideal, then I^e need not be a prime ideal of B.

7. A subgroup H of a finite group G is called a *Hall subgroup of* G if its index in G is relatively prime to its order: $(|G : H|, |H|) = 1$.

   (a) Recall that the Second Isomorphism Theorem for Groups says:

   Let G be a group, let A and B be subgroups of G and assume $A \leq N_G(B)$. Then AB is a subgroup of G, $B \trianglelefteq AB, A \cap B \trianglelefteq A$ and $AB/B \cong A/(A \cap B)$.

   Prove this theorem. (You may assume the First Isomorphism Theorem for Groups.)

   (b) Use the Second Isomorphism Theorem for Groups to prove that if H is a Hall subgroup of G and $N \trianglelefteq G$, then $H \cap N$ is a Hall subgroup of N.