Algebra Qualifying Exam, problem set

August 22, 2014

Do all seven problems. Each problem is ten points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let G and H be subgroups of the permutation group S_8.

   (a) Let G be the group generated by permutations $\sigma = (1\,4\,2\,5\,3)(6\,7\,8)$ and $\tau = (1\,2\,3\,4\,5)$. Determine the order of G, and whether or not G is a cyclic group. If G is a cyclic group, find a generator; if not, give a complete proof.

   (b) Let H be the group generated by permutations $\eta = (1\,2\,3\,4\,5)(6\,7)$ and $\tau = (1\,2\,3\,4\,5)$. Answer the same questions as in Part (a).

   (c) Recall the Dihedral group $D_{10} = \langle r, s | r^5 = s^2 = 1, rsr^{-1} = s\rangle$. We also know that D_{10} can be generated by $\beta = rs$ and r. Is H in Part (b) isomorphic to D_{10}? If they are isomorphic, construct an isomorphism and justify your construction. If not, justify your conclusion.

2. Let G be a group. We recall the definition of the center $Z(G)$ of G:

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G.\}$$

   (a) Prove that $Z(G)$ is a normal subgroup of G.

   (b) Prove that if $G/Z(G)$ is cyclic, then G is Abelian.

   (c) Assume the order of G is pq where p and q are primes. Assume also that $Z(G)$ is a nontrivial subgroup of G. Prove that $Z(G) = G$.

3. Let k be a field, and let $R = k[x]/(x^5)$ be the quotient of the polynomial ring $k[x]$ in one variable by the principal ideal generated by x^5.

   (a) Find all units in the ring R.

   (b) Show that R has a unique maximal ideal.

4. Let $R = \mathbb{Z}[\sqrt{-13}]$.

   (a) Show that 7 is irreducible in the ring R.

   (b) Show that 7 is not a prime in the ring R.

   (c) Give an example of an element of R which admits two different factorings into irreducibles.

5. Let $f(x) = x^3 - 2$. Find the Galois group of f

   (a) over $K_1 = \mathbb{Q}(\sqrt[3]{2})$.

   (b) over $K_2 = \mathbb{Q}(\sqrt{-3})$.

   (c) over $K_3 = \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$.

6. (a) Prove that if G is a group and $|G| = 2^2 \cdot 5 \cdot 19$, then G has a normal Sylow p-subgroup for either $p = 5$ or $p = 19$.

   (b) Let K be a Galois extension of $\mathbb{Q}$. Prove: If $[K : \mathbb{Q}] = 2^2 \cdot 5 \cdot 19$, then there exists some subfield E of K such that E is Galois over $\mathbb{Q}$ and $[K : E]$ is prime.

7. Let $\alpha = \sqrt{3} + \sqrt{5}$.

   (a) Prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$.

   (b) Show that $\mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and find its Galois group.

   (c) Find all subfields of the field $\mathbb{Q}(\alpha)$.