Algebra Qualifying Exam

August 2015

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let R and S be commutative rings with 1. Suppose that I and J are ideals of R and S respectively. **Prove**: If $\psi : R \to S$ is an isomorphism with $\psi(I) = J$, then R/I is isomorphic to S/J.

2. Let R be a ring, not necessarily commutative. We say that R is *Boolean* if $a^2 = a$ for all $a \in R$.

   (a) **Prove**: If R is Boolean, then $2x = 0$ for all $x \in R$.

   (b) **Prove**: Every Boolean ring is commutative.

   (c) Find all Boolean rings that are integral domains.

3. (a) Assume that the group H has order p where p is a prime. Prove that every nonidentity element has order p.

   (b) Let G be a group, and let H and K be subgroups of G of prime order p. Prove that either $H = K$ or $H \cap K$ is the trivial subgroup $\{1\}$.

   (c) How many elements of order 7 are there in a simple group of order 168?

4. (a) Show that there is a nonabelian subgroup T of $S_3 \times Z_4$ of order 12 generated by elements a, b such that $|a| = 6$, $a^3 = b^2$, $ba = a^{-1}b$.

   (b) Prove that no two of D_{12}, A_4 and T are isomorphic, where T is as defined in Part (a). Note that D_{12} is the dihedral group of order 12.

5. (a) Show that a polynomial of degree 3 over a field is either irreducible or has a root in the field. Is $x^3 + 5x + 1$ irreducible over $\mathbb{Q}$?

   (b) Find the Galois group of $x^3 + 5x + 1$ over $\mathbb{Q}$.

6. Compute the Galois group of the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

7. Suppose K is an extension of degree 7 over $\mathbb{Q}$. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial with no repeated roots.

   (a) Assume that all roots of $f(x)$ are in K, but not all of the roots are in $\mathbb{Q}$. Prove that K is the splitting field of $f(x)$.

   (b) Give an explicit example to show that Part (a) is false if $[K : \mathbb{Q}] = 6$; instead of 7.