Algebra Qualifying Exam
August 26, 2016

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let G be a group. **Prove**: If $|G| = 4 \times 3^{2016}$, then G is not simple.

2. Let $\alpha = \sqrt[3]{3} + \dfrac{1}{\sqrt[3]{3}}$. Answer parts (a) through (c), and justify your answer in each case.
   (a) Find the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.
   (b) Find the minimal polynomial of α over $\mathbb{Q}$.
   (c) Is the extension $\mathbb{Q}(\alpha)$ Galois?

3. (a) **Prove**: A finite integral domain is a field.
   (b) **Prove**: If R is a finite commutative ring with 1, then every prime ideal of R is maximal.

4. Let
$$V = \{1, (12)(34), (13)(24), (14)(23)\}.$$
   Note that V is a subgroup of S_4. (You may assume this without proof.)
   (a) Explain why V is a normal subgroup of S_4.
   (b) **Prove**: If $\sigma, \tau \in S_4$ and $\sigma\tau^{-1} \in V$, then either $\sigma = \tau$ or 4 is not a fixed point of $\sigma\tau^{-1}$.
   (c) **Prove**: $S_4/V \simeq S_3$. (Note: If you use the classification of groups of order 6, you should prove it.)

5. (a) **Prove**: The quotient ring $\mathbb{Z}[x]/(2, x^3 + 1)$ has exactly 8 elements.
   (b) Find all ideals of the quotient ring $\mathbb{Z}[x]/(2, x^3 + 1)$. Justify your answer answer.
   (c) Is $\mathbb{Z}[x]/(2, x^3 + 1)$ a field? Explain why or why not.

6. Let $f(x) = x^4 + ax^2 + b \in \mathbb{Z}[x]$ be irreducible. Let G be the Galois group of $f(x)$. Note that f is an even function, so we may write the roots of f as $\alpha, \beta, -\alpha, -\beta$.
   Prove: If $\mathbb{Q}(\alpha\beta) = \mathbb{Q}(\alpha^2)$, then G is cyclic of order 4. **Hint**: First show that $\beta \in \mathbb{Q}(\alpha)$.

Continued on the other side.

7. Recall that if H is a subgroup of a group G, then

$$N_G(H) = \{g \in G : g^{-1}Hg = H\} \text{ and } C_G(H) = \{g \in G : g^{-1}hg = h \text{ for all } h \in H\}.$$

Let p be a prime, P be a subgroup of S_p and $|P| = p$.

(a) Prove that $|C_{S_p}(P)| = p$.

(b) Prove that $|N_{S_p}(P)| = p(p-1)$.