Algebra Qualifying Exam

January 13, 1995

There are three parts. In the first part you are to do 3 of 4 problems. In the second you are to do 6 of 7 problems. In the third part are to do 3 of 5. Each part is worth 30 points; a grade of 65 out of 90 is expected for passing.

Please do each problem on separate sheets.

In every problem you *must* give evidence of your reasoning.

The bold face letters **C, R, Q, Z** stand for the complex, real, rational numbers, and integers respectively.

Part I (30 points.) Do three of four. Each problem is worth 10 points. (If you do 4 problems, only the first 3 will be graded.)

1 Let T be a linear transformation from a vector space V to a vector space W. Let B denote a basis for V. Show that $T(B)$ is a basis for the image of T if and only if T is one-to-one.

2 (a) Define what it means to say that a linear transformation T from a vector space V into V is nilpotent of index k.

(b) Assume T is nilpotent of index k on V. Show that there is a vector $u \in V$ so that the set $B = \{u, Tu, T^2u, ..., T^{k-1}u\}$ is linearly independent.

(c) If W is the linear span of the set $\{u, Tu, T^2u, ..., T^{k-1}u\}$ of part (b), find the matrix A which represents the restiction T_1 of T to W with respect to the basis B.

(d) What would be the matrix A in part (c) if we wrote the basis in the order $\{T^{k-1}u, T^{k-2}u, ..., Tu, u, \}$?

3 (a) Prove that if A and B are similar matrices, then A and B have the same characteristic polynomials and the same minimal polynomials.

(b) Give an example of two non-similar matrices A, B with minimal polynomial $m(x) = (x+3)^2(x-5)^3$ and characteristic polynomial $c(x) = (x+3)^4(x-5)^3$. If this is not possible, explain why.

4 Let V be a vector space with inner product $<,>$ and let S be a subspace of V. The orthogonal complement $S^\perp$ is defined as the set $\{x \in V :< x, s >= 0 \; \forall s \in S\}$.

(a) Show that the orthogonal complement of a subspace is a subspace.

(b) Let S be a subspace of $\mathbf{C}^4$ with orthonormal basis $B = \{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}), (\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The vector $u = (3, 25, 79, 101)$ is a member of S. Write u as a linear combination of the vectors in B.

(c) Compute $S^\perp$ where V and S are from part (b).

(d) Prove in general that if V is a vector space with inner product $<,>$ and subspace S then $V = S \bigoplus S^\perp$.

Part II (30 points.) Do six of seven. Each problem is worth 5 points. (If you do 7 problems, only the first 6 will be graded.)

1 (a) How many Sylow 5-subgroups does S_5 (the symmetric group on 5 symbols) have?
(b) List the elements in one of them.

2 State and prove Cayley's theorem.

3 (a) Give an example of an abelian simple group.
(b) Give an example of a nonabelian simple group.

4 Let M_{nn} denote the set of all $n \times n$ matrices with real number entries. Name an operation $*$ on M_{nn}, a group $(S, *)$ where S is a subset of M_{nn} containing more than one matrix, and a nontrivial group $(G, \circ)$ so that
(a) The determinant function is a homomorphism from $(S, *)$ onto $(G, \circ)$.
(b) The transpose function $(A \longrightarrow A^t)$ is a homomorphism from $(S, *)$ onto $(G, \circ)$.
(Any of $S, *, G$, and $\circ$ may be different in (b) than those you created in (a).)

5 (a) Describe all elements of the group $\mathbf{Q}/\mathbf{Z}$ that have finite order.
(b) Describe all elements of the group $\mathbf{R}/\mathbf{Q}$ that have finite order.

6 Let G be a group of order 95 and suppose f is a homomorphism from G onto a group H. Suppose also that f is not an isomorphism. Prove that H is cyclic.

7 A group is solvable iff there exist subgroups

$$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq ... \supseteq N_r = \{e\}$$

such that each N_i is normal in N_{i-1} and N_i/N_{i-1} is abelian. Prove that a subgroup of a solvable group is solvable.

Part III (30 points.) Do three of five. Each problem is worth 10 points. (If you do more than three problems, only the first 3 will be graded.)

1 Set $\Psi_p(x) := \frac{x^p-1}{x-1}$ where p is a prime.

(a) Prove using Eisenstein's criterion that $\Psi_p(x)$ is irreducible over $\mathbf{Q}$.

(b) Is $\Psi_p(x)$ is irreducible over $\mathbf{Z}$?

(c) Prove or disprove: If $a(x) + <\Psi_p(x)>$ is a nonzero member of the quotient ring $\mathbf{Q[x]/<\Psi_p(x)>}$ then $a(x) + <\Psi_p(x)>$ is a unit.

2 Let G' represent the derived subgroup of a group G, that is, the group generated by the commutators of G.

(a) Prove that if N is a normal subgroup of G such that G/N is abelian then $G' \subseteq N$

(b) Prove that if ϕ is a homomorphism into the multiplicative group of nonzero complex numbers then the kernel of ϕ contains G'.

(c) Compute G' when G is the dihedral group $< a, b : a^4 = b^2 = 1, bab^{-1} = a^3 >$ of order eight and classify the abelian group G/G'.

3 Let $\zeta = e^{\frac{2\pi i}{13}}$ so that $\zeta^{13} = 1$. Let $\theta = 1 + \zeta^3 + \zeta^9$.

(a) Explicitly describe the Galois group $G = Gal(\mathbf{Q}(\zeta)/\mathbf{Q})$ and its subgroups.

(b) Identify the subgroup of G which fixes $\mathbf{Q}(\zeta)$.

(c) Find the dimension of $\mathbf{Q}(\theta)$ over $\mathbf{Q}$ and give a basis for $\mathbf{Q}(\theta)$ over $\mathbf{Q}$.

(d) Construct the irreducible polynomial of θ over $\mathbf{Q}$.

4 (a) Let $f(x)$ and $g(x)$ be polynomials over an integral domain D. Suppose $GCD(f(x), (g(x)) = h(x)$. Prove the following version of the Chinese Remainder Theorem:

The pair of simultaneous congruences

$$t(x) \equiv a(x) \mod f(x)$$

$$t(x) \equiv b(x) \mod g(x)$$

has a solution modulo $\frac{f(x)g(x)}{h(x)}$ if and only if $h(x)$ divides $(a(x) - b(x))$.

(b) Solve the following pair of simultaneous congruences over the polynomial ring $\mathbf{Q}[x]$:

$$t(x) \equiv x^4 \mod (x^4 + 2x^3 - 3x - 6)$$

$$t(x) \equiv x^2 - 10x - 8 \mod (x^3 + 2x^2 + x + 2)$$

5 For this problem, F is the field of order 3. You are given that f and g are irreducible polynomials over F of degrees 4 and 2 respectively. Set $R := F[x]/ < f \cdot g >$.

(a) Prove or disprove R is a field.

(b) Prove or disprove R is an integral domain.

(c) Find the cardinality of R.

(d) Classify the group of units of R as a direct product of abelian groups.