Algebra Qualifying Exam

January 18, 1997

There are three parts. Part A covers linear algebra, part B covers group theory, and part C covers ring theory. To pass the exam, satisfactory performance on each of the three parts is expected.

In some places on this test you are instructed to do a certain number of questions from a larger number of questions. If you do more than the stated number of parts, only the first ones numerically or alphabetically will be scored.

Part A (60 points) In part 1, do any <u>two</u> (2) of a, b, or c. In part 2, do any <u>four</u> (4) of a-f. Each of the six parts is worth 10 points.

1. Do any <u>two</u> (2) of a, b, or c.

   a. Consider two vector spaces V, W, both over a field F and of dimension 2. Show that there is an isomorphism between V and W.

   b. Let V be the vector space of all polynomial functions p from $\mathbf{R}$ into $\mathbf{R}$ which have degree less than or equal to 1. Let t_1 and t_2 be distinct real numbers and let $L_i: V \to \mathbf{R}$ be defined by $L_i(p) = p(t_i)$ for i = 1,2.

      i. Show that L_1 and L_2 are independent linear functionals on V. Do they form a basis for V*, the dual of V? Briefly support your answer.

      ii. Find explicitly the basis for V of which $\{L_1, L_2\}$ is the dual basis.

   c. Let V be a finite dimensional vector space of dimension n. Show that any subset of V which contains more than n vectors is linearly dependent.

2. Do any <u>four</u> (4) of parts a-f on the next page.

a. i. Find the row-reduced echelon form of the following matrix.

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & -2 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

ii. What is the column rank of the above matrix?

b. We know that F[-1,1], the set of all real-valued functions defined on [-1,1], is a vector space over **R**. Let V = C[-1,1], the set of continuous functions on [-1,1], and $L(f) = \int_{-1}^{1} f(t)\,dt$. Show that V is a vector space over **R** and show that L is a linear functional (linear transformation) from V to **R**.

c. Suppose that V is a two dimensional vector space over **C**. Let $T \in L(V,V)$, the set of all linear transformations from V to V. Suppose that the polynomial $p(\lambda) = \lambda^4 + 1$ annihilates T. Is T diagonalizable? Support your answer.

d. Let D^2 be the second derivative operator on the set V of real valued quadratic polynomials. Find a matrix representation for D^2 and the transpose $(D^2)^t$. What is the null space of D^2? What is the null space of $(D^2)^t$?

e. What is the subspace of $\mathbf{R}^3$ annihilated by the set of linear functionals which follow?

$$\begin{aligned} f_1(x_1,x_2,x_3) &= x_1 + 2x_2 + 2x_3 \\ f_2(x_1,x_2,x_3) &= 2x_2 \\ f_3(x_1,x_2,x_3) &= -2x_1 - 4x_3 \end{aligned}$$

f. Find the coordinates of the vector (1,1,1) in the subspace spanned by the set of vectors $\{\alpha_1,\alpha_2,\alpha_3\}$ such that the matrix P with columnwise entries α_1, α_2, and α_3 ($P = [\alpha_1\ \alpha_2\ \alpha_3]$) has inverse

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Part B (60 points) Answer all five questions in this part.

1. Prove that the set theoretic union of two subgroups of a group is a subgroup if and only if one of the subgroups is contained in the other.

2. Let $f: G \to H$ be a homomorphism of groups. Suppose K is the kernel of f. Prove the following:

   a. K is a subgroup of G.
   b. K is a normal subgroup of G.
   c. The quotient group G/K is isomorphic to the image of f.

3. Let GL(n,**R**) denote the group of all n x n matrices over **R** whose determinant is not zero. Let SL(n,**R**) = {A: A $\in$ GL(n,**R**), det A = 1}. Prove that SL(n,**R**) is a normal subgroup of GL(n,**R**) and that GL(n,**R**)/SL(n,**R**) is isomorphic to the group of non-zero elements of **R** under multiplication.

4. a. State precisely the full content of the Sylow Theorems for a finite group G whose order is divisible by a prime p.
   b. Prove that a group G of order 126 must have a normal subgroup of order 7.
   c. Prove that a group G of order 1000 cannot be simple, i.e., G must have a normal subgroup N such that $N \neq \{e\}$ and $N \neq G$.

5. Let G = <a> be a cyclic group of order n. Prove that G = <a^k> if and only if gcd(k,n) = 1. Deduce that an integer k is a generator of $\mathbf{Z}_n$ if and only if gcd(k,n) = 1.

Part C (60 points) Answer any five (5) of parts 1-7 below. Each is worth 12 points.

1. Suppose R is a ring and n is a positive integer. Are there any combinations of R and n for which $(x + y)^n = x^n + y^n$ for every $x,y \in R$? If so, state general conditions for which this is true. If not, prove why not.

2. Suppose $f: R \to S$ is a homomorphism from a ring R to a ring S. Suppose J is an ideal of S. Show that $f^{-1}(J)$ is an ideal of R.

3. Prove that $\mathbf{R}[x]/\langle x^2 + 1\rangle \cong \mathbf{C}$ where $\langle x^2 + 1\rangle$ denotes the ideal generated by $x^2 + 1$. Show they are isomorphic as rings.

4. Suppose R is a ring and S is a subring of R.

   a. Which one of the following is true?
      i. If M is an R-module, then M is an S-module.
      ii. If M is an S-module, then M is an R-module.

   b. Prove it.

5. Let F be a field. Do <u>one</u> (1) of the following.

   a. State and prove necessary and sufficient conditions for f(x) to be invertible where $f(x) \in F[x]$.

   b. State and prove necessary and sufficient conditions for f(x) to be invertible where $f(x) \in F[[x]]$.

6. Prove this part of the isomorphism theorems:

   If S is a subring of R and I is an ideal of R, then $(S + I)/I \cong S/(S \cap I)$.

7. Prove <u>either</u> the "if" part <u>or</u> the "only if" part of the following theorem.

Let R be a commutative ring with identity. Let I be an ideal of R. Then I is a maximal ideal of R if and only if R/I is a field.