Algebra Qualifying Exam, January 2002

There are thirteen problems on three pages. The total point value of the exam is 120 points.

1. (10 points)

   (a) Consider the vector space $P_2(\mathbf{R})$ of all polynomials of degree two or less, over the field of real numbers $\mathbf{R}$ What is the coordinate vector of $p(x) = (x+1)^2$ with respect to the standard basis $\mathrm{B} = \{1, x, x^2\}$?

   (b) Which of the following sets are also bases for $P_2(\mathbf{R})$? (Name all that are.)

      i. $C = \{1+x,\ 1+x^2\}$
      ii. $D = \{1+x,\ 1+x+x^2\}$
      iii. $E = \{1+x,\ 1+x^2,\ 2+x+x^2\}$
      iv. $F = \{1-x,\ 1+x, x^2\}$
      v. $G = \{1,\ 1+x,\ 1+x^2,\ 1+x+x^2\}$
      vi. $H = \{1+x,\ 1+x^2,\ x+x^2,\ 1+x+x^2\}$

   (c) Find the coordinate vector of $p(x)$ with respect to one of the bases you found in (b). (Make sure the grader knows which basis you are choosing.)

2. (5 pts.) Suppose V is an n-dimensional vector space over $\mathbf{R}$, the set of reals, where n is a positive integer. Must V contain an m dimensional subspace for every integer m with $0 \leq m \leq n$? Why or why not?

3. (5 pts.) Let V be an inner product space and suppose $T : V \to V$ is a self-adjoint transformation. Prove that if $\vec{v}$ and $\vec{w}$ are eigenvectors of T with different eigenvalues then $\vec{v}$ and $\vec{w}$ are orthogonal. (Hint: consider $< T\vec{v}, \vec{w} >$.)

4. (5 pts.) The following set of integers $\{1, 9, 16, 22, 53, 74, 79, 81, a\}$ forms a group under multiplication modulo 91. What is a?

5. (10 pts.)

   (a) Let a be an element of a finite group G, and $|a| = n$. Show that $|a^k| = |a^{n-k}|$, for $0 \leq k \leq n$

   (b) Suppose b is a 10-cycle permutation. For which integers i between 2 and 10 is b^i also a 10-cycle?

   (c) Let a, and b be elements of a group G such that $|a| = m$ and $|b| = n$ and $\gcd(m, n) = 1$. Show that $< a > \cap < b > = 1$.

6. (15 pts.) $T : V \to W$ be a module homomorphism.

   (a) Define: *null space* of T. (An alternate word for "null space" is "kernel".)

   (b) Prove the null space is a submodule of V.

   (c) Prove that if S is a subspace of W then $T^{-1}(S)$ is a subspace of V.

7. (5 pts.) Let $\mathbf{P}$ be the set of polynomials over the reals $\mathbf{R}$. Given $a \in \mathbf{R}$, define $g_a : \mathbf{P} \to \mathbf{R}$ by $g_a(p(x)) = p(a)$. Show that g_a is in $\mathbf{P}^*$, the dual space of P.

8. (10 pts.) Suppose G is a nonabelian group of order $4p$, p a prime, and suppose that G has a normal subgroup of order 4. Prove that $p = 2$ or $p = 3$ and identify G in these cases.

9. (10 pts.) Let $GL(n, R)$ denote the set of all $n \times n$ matrices over a ring R whose determinant is different from zero. Set $SL(n, R) = \{A : A \in GL(n, R) \text{ and } \det(A) = 1\}$.

   You may assume that $GL(n, R)$ is a group under matrix multiplication.

   (a) Prove that $SL(n, R)$ is a normal subgroup of $GL(n, R)$.

   (b) Prove that $GL(n, R)/SL(n, R)$ is isomorphic to the group of nonzero elements of R under multiplication.

10. (5 pts.) A normal series of a group G is a chain (*) of subgroups

$$(*) \qquad G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots \supseteq N_r = \{1\}$$

   such that each N_i is normal in N_{i-1}. The factor groups of (*) are the groups N_{i-1}/N_i.

   Find a normal series for S_4 of maximal length.

11. (15 pts.) Let $\zeta := e^{\pi i/4}$. Let $\mathbf{Q}$ be the field of rationals and $E := \mathbf{Q}(\zeta)$.

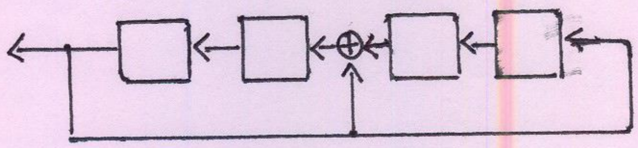
   (a) Compute the minimal polynomial of ζ.

   (b) Compute the Galois group of E over $\mathbf{Q}$.

   (c) Find all subfields of E.

   (d) Show the Galois correspondence between the subfields of E and the subgroup of the Galois group computed in part (b).

12. (10 pts.) Consider the linear feedback shift register below, acting on members of Z_2, the field of order 2.



(a) Describe the associated quotient ring.

(b) Is the associated quotient ring a field? (If not, why not?)

(c) Suppose the initial state of the shift register is 0 0 0 1. Find the state of the shift register 662 clock cycles later.

13. (15 pts.) A **local ring** is a commutative ring with unity which has a unique maximal ideal.

(a) Give an example of a finite local ring which is not a field.

(b) Give an example of an infinite local ring which is not a field.

(c) Prove that in a local ring R with maximal ideal M, the set of units is precisely those elements of R which are not in M.

(d) Give an example of a ring with unity, which has a maximal ideal M, yet has the property that there exist nonunits not in M.