Algebra Qualifying Exam

January 10, 2003

Calculators are not allowed.

Each part has five problems. In each part you are to do the first problem and three of the remaining four. Please specify which problems you are doing. (If you do all the problems in a single part then problem 5 will not be graded.)

Part I. Linear Algebra

1. Let $T : V \to W$ be a one-to-one linear transformation from V *onto* W. Prove that if $\{b_1, b_2, \ldots, b_n\}$ is a basis of V then $\{T(b_1), T(b_2), \ldots, T(b_n)\}$ is a basis of W.

2. A square matrix, A, is called *nilpotent* if $A \neq 0$ but $A^n = 0$ for some positive integer n.

   (a) Prove that if A is invertible, then A is not nilpotent.

   (b) If A is square but not invertible must A be nilpotent? Justify your answer.

   (c) Show that if A and B are nilpotent matrices and $AB = BA \neq 0$ then AB is nilpotent.

   (d) Show that if A and B are nilpotent matrices and $AB \neq BA$ then AB is not necessarily nilpotent.

   (e) Show that if A and B are nilpotent matrices, $AB = BA$, and $A + B \neq 0$ then $A + B$ is nilpotent.

   (f) Show that if A and B are nilpotent matrices, $AB \neq BA$, and $A + B \neq 0$ then $A + B$ is not necessarily nilpotent.

   (g) Show that 0 is the only eigenvalue of a nilpotent matrix.

3. Let

$$A = \begin{pmatrix} 6 & 3 & 6 \\ 1 & 4 & 2 \\ -2 & -2 & -1 \end{pmatrix}$$

   where A is considered as a matrix over the real numbers.

   (a) Find the minimal polynomial of A.

   (b) Find the eigenvalues of A and the eigenspaces belonging to these eigenvalues.

   (c) Is A diagonable? Why or why not?

4. Let

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 & -2 \\ 0 & -1 & -3 & 0 & 3 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -9 & -1 & 9 \\ -1 & 0 & -2 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Then B is similar to A.

(a) If T denotes the linear transformation defined as left multiplication by A, find the T-cyclic subspace generated by $v = \langle 1, 1, 0, 0, 0 \rangle$

(b) Find a matrix Q such that $A = Q^{-1}BQ$.

(c) Find the characteristic and minimal polynomials of B.

5. (a) Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

(b) Let V be a vector space over the reals and W an inner product space also over the reals. If $T : V \to W$ is linear, prove that $(x, y)' = (T(x), T(y))$ defines an inner product on V if and only if T is one-to-one.

Part II. Group Theory

1. Suppose G is a group of order pq, where p and q are distinct primes. Suppose there is a homomorphism from G onto a group H. Prove that H is isomorphic to G or H is cyclic.

2. (a) Give an example of a Sylow 2-subgroup of A_5, call it K.

(b) Prove that K has exactly 5 conjugates in A_5.

3. Suppose H is a subgroup of group G of index 2. Show that $a^2 \in H$ for every $a \in G$.

4. A special case of the Fundamental Theorem of Finite Abelian Groups is that an abelian group of order p^2, where p is a prime, is either cyclic or isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Prove this special case.

5. Suppose that X is a set and G is a group. An action of G on X is a function $\alpha : G \times X \to X$, denoted by $\alpha : (g, x) \mapsto g \cdot x$, such that

(i) $1 \cdot x = x$ for all $x \in X$, and

(ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and for all $x \in X$.

For $x \in X$, we let $G_x = \{g \in G : g \cdot x = x\}$. Let $x, y \in X$ such that $y = g \cdot x$ for some

Part III. Ring Theory

1. Let R be a commutative ring with unity and M a maximal ideal of R. Prove

   (a) M is a prime ideal.

   (b) R/M is a field.

2. Let $\mathbb{Z}_n$ represent the ring of integers modulo n.

   (a) Prove that $a \in \mathbb{Z}_n$ is a unit of $\mathbb{Z}_n$ if and only if a relatively prime to n.

   (b) Find the inverse of 111 in $\mathbb{Z}_{11131}$.

3. (a) Let M be a subset of a ring R. Prove that M (under the inherited operations of R) is a left R-module if and only if M is a left ideal of R.

   (b) Let

$$A = \begin{pmatrix} 6 & 3 & 6 \\ 1 & 4 & 2 \\ -2 & -2 & -1 \end{pmatrix}$$

   be the matrix of problem 2 in the Linear Algebra portion and let S be the subring of $Mat_3(\mathbb{R})$ generated by A. Is the set $\{\begin{pmatrix} a \\ a \\ -a \end{pmatrix} : a \in \mathbb{R}\} \subseteq \mathbb{R}^3$ an S-module?

4. Let F be a field and $F[x]$ a ring of polynomials over F and f a member of $F[x]$. Prove that $< f >$ is a maximal ideal if and only if f is irreducible.

5. Let F be the field with two elements.

   (a) Prove that $f_1 := x^4 + x + 1$ and $f_2 := x^4 + x^3 + x^2 + x + 1$ are irreducible elements in $F[x]$.

   (b) Find the order of $x + \langle f_i \rangle$ in $F[x]/\langle f_i \rangle$ (for $i = 1, 2$).

   (c) Construct a binary linear feedback shift register on 4 bits which has period 15.