January 2006
Algebra Qualifying Exam

There are 90 points possible on this exam (each problem is worth 10 points and you must do three problems in each section), a passing grade is typically 70 percent (63 points). Partial credit will be given sparingly - getting entire problems correct is better than amassing partial credit. Be aware that points will be taken for wrong statements.

MTH 525 - Linear Algebra

Do 3 of the 4 following problems.

(1) Let $A = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.
   (a) Find the minimal polynomial of A.
   (b) Find the rational form of A.
   (c) Find the Jordan form of A.

(2) Let V_1, V_2, V_3 be finite dimensional vector spaces over a field F with $d_i = \dim(V_i)$. Suppose there are linear transformations $T : V_1 \to V_2$ and $S : V_2 \to V_3$ such that range$(T) =$ nullspace(S).
   (a) Show that $d_1 + d_3 \geq d_2$.
   (b) Show that $d_1 + d_3 = d_2$ if and only if T is one-to-one *and* S is onto.

(3) Let A, B be $n \times n$ matrices over a field F, and suppose that there exists a matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices (we say A and B are *simultaneously diagonalizable*). Prove that $AB = BA$.

(4) Let A be an $n \times n$ matrix with complex entries, and suppose that A^m is the zero matrix for some $m \in \mathbb{N}$.
   (a) Show that A^n is the zero matrix.
   (b) Show that A is diagonalizable if and only if A itself is the zero matrix.
   (c) If A^2 is the zero matrix, show that $\det(I + A) = 1$.

MTH 623 - Group Theory

Do 3 of the 4 following problems.

(1) Let G be a finite group. Let H be a subgroup of G of index p, where p is a prime. Prove that if p is the smallest prime dividing the order of G, then H is normal.

(2) Prove that there does not exist a simple group of order 36.

(3) Let G a finite group of order greater than 1. Prove that if G is solvable then it has a normal abelian subgroup. [Hint: Consider minimal normal subgroups].

(4) Let G be a finite group, and suppose that its center, $Z(G)$, is trivial. Prove that the order of G divides the order of $\mathrm{Aut}(G)$, the automorphism group of G.

MTH 625 - Ring Theory

Do 2 of the first 3 problems, and do number 4. All rings are assumed to be commutative, with unity.

(1) Let R be a ring.
   (a) Given a subset $S \subseteq R$ of a ring R, define the ideal generated by S.
   (b) Show that $\langle a, b\rangle$, the ideal generated by the pair of elements a and b, is equal to the set $\{ar_1 + br_2 : r_1, r_2 \in R\}$.
   (c) Suppose a and b are ring elements where the principle ideals $\langle a\rangle$, $\langle b\rangle$ satisfy
   $$\langle a\rangle + \langle b\rangle = R.$$
   Show that, for any ring element c,
   $$\langle a, c\rangle \cdot \langle b, c\rangle = \langle ab, c\rangle .$$

(2) Let R and S be rings and let $f : R \to S$ be a ring homomorphism which is onto S. Prove:
   (a) if P is a prime ideal of R then $f(P)$ is a prime ideal of S.
   (b) if I is an ideal of R and P is a prime ideal of R containing I then P/I is a prime ideal of R/I.
   (c) if I is an ideal of R then every prime ideal of R/I is of the form P/I where P is a prime ideal of R containing I.

(3) Here $\mathbb{Z}$ represents the ring of integers. Show:
   (a) the ring of polynomials, $\mathbb{Z}[x]$, is a free $\mathbb{Z}$-module.
   (b) the field of rationals, $\mathbb{Q}$, is *not* a free $\mathbb{Z}$-module.

(4) Let $\zeta := e^{\frac{2\pi i}{7}}$ be a seventh root of unity and $\mathbb{Q}(\zeta)$ the field generated by the rationals and ζ.
   (a) Give the minimal polynomial for ζ over $\mathbb{Q}$.
   (b) Describe precisely the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
   (c) List the normal subgroups of the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ and the corresponding subfields of $\mathbb{Q}(\zeta)$. (For each subfield F, $\mathbb{Q} < F < \mathbb{Q}(\zeta)$, describe F by providing a basis for F over $\mathbb{Q}$.)