Algebra Qualifying Exam

January 12, 2007

Each problem is worth ten points. A score of 65 (out of 90) is passing.

1. Let A be an $n \times n$ matrix with entries in a field F of characteristic 0. Prove that A is invertible if and only if there is a polynomial $p(x)$ such that

   (a) $p(0) = 0$

   (b) $p(A) = I$.

2. Let V be a vector space over a field F. Prove that V has a basis. (Do not assume that V is finite dimensional.)

3. Let A, B be 3×3 complex matrices. Show that if A and B have the same characteristic and minimal polynomials, then A and B are similar.

4. Suppose G is a group of order 20 with an element g of order 4. Construct up to isomorphism all groups with this property.

5. Consider the following property, let's call it Property T, for "transitive": for any pair of nonidentity elements x, y in a group G there is an automorphism ϕ of G such that $\phi(x) = y$.

   (a) Show that there is a group of order 27 with property T.

   (b) Show that there is only one group of order 27 with property T.

   (c) Find all finite groups with this property.

6. Suppose that G is a simple group with a subgroup H of index n. Prove that the order of G must divide $n!$.

7. Recall that a ring is artinian if the descending chain condition holds on ideals. Assume that R is a commutative artinian ring with unity. Prove that every prime ideal in R is maximal.

8. Show that there exists a Galois extension K of $\mathbb{Q}$, the field of rational numbers, whose Galois group G is cyclic of order 7. (Hint: consider cyclotomic extensions.)

9. Recall that a field K is algebraically closed if every polynomial $f(x) \in K[x]$ splits in K. Let $F \subseteq E$ be an algebraic field extension. Prove that if every polynomial $f(x) \in F[x]$ splits in E, then E is algebraically closed.