Algebra Qualifying Exam
January 3, 2008

Do all six problems. The exam is 50 points. A passing grade is 35/50.
No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. (10 points)
   Let $k \geq 1$ be an integer. **Prove**: No group of order $2^k \cdot 5$ is simple. (**Note:** If you use Burnside's Theorem, you should prove it. Alternately, if you use the Classification of Finite Simple Groups, you should prove it.)

2. (5 points)
   Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Suppose that E is a splitting field for F and assume that there exists an element $\alpha \in E$ such that both α and $\alpha + 1$ are roots of $f(x)$. Show that the characteristic of F is not zero.

3. (5 points)
   Let X be a a subspace of $M_n(\mathbb{C})$, the $\mathbb{C}$-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in X is invertible. Prove that $\dim_{\mathbb{C}} X \leq 1$.

4. (10 points)
   Let V be a finite-dimensional vector space over an algebraically closed field F, and let S and T be commuting linear operators on V. Assume that the characteristic polynomial of S has distinct roots.
   (a) (5 points) **Prove**: Every eigenvector of S is an eigenvector of T.
   (b) (5 points) **Prove**: If T is nilpotent, then $T = 0$.

5. (10 points)
   Let $\mathbb{Q}$ be the field of rational numbers, and let $f(x) = x^8 + x^4 + 1$ be a polynomial in $\mathbb{Q}[x]$. Suppose F is a splitting field for $f(x)$ over $\mathbb{Q}$ and set $G = \mathrm{Aut}_{\mathbb{Q}} F$.
   (a) (5 points) Find $[F : \mathbb{Q}]$, and determine the Galois group G up to isomorphism.
   (b) (5 points) If $\Omega \subseteq F$ is the set of roots of $f(x)$, find the number of orbits for the action of F on Ω.

6. (10 points)
Let M be a $\mathbb{Z}$-module; i.e., M is an abelian group. Suppose that

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

is a chain of submodules such that, for $i = 1, 2, \ldots, n$, the factors M_i/M_{i-1} are simple and pairwise non-isomorphic.
Prove: If X and Y are isomorphic submodules of M, then $X = Y$.