Algebra Qualifying Exam
January 9, 2009

Do all six problems. The exam is 60 points. A passing grade is 42/60.
No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. (12 pts) Suppose G is a finite group and that the order of G is a product of three distinct primes. Prove that G has a normal p-Sylow subgroup for some prime p.

2. (8 pts) Let G be a group and let H be a subgroup of G.

   (a) **Prove**: If H is normal in G and $[G : H] = n$, then $g^n \in H$ for all $g \in G$.

   (b) Give an example to show that the conclusion of part (a) is false if the hypothesis "*H is normal in G*" is omitted.

3. (10 pts) Let

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 5 & -2 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

   Find the characteristic polynomial, the minimal polynomial, and the Jordan canonical form for A. Find an invertible 3×3 matrix P such that $P^{-1}AP$ is the Jordan canonical form.

4. (10 pts)

   (a) Suppose that $T : \mathbb{C}^2 \to \mathbb{C}^2$ is a linear transformation, and that there is some basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$ of $\mathbb{C}^2$ such that the matrix of T with respect to $\mathcal{B}$ is

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}.$$

   Let $\mathcal{B}' = \{\alpha_2, \alpha_1\}$. Find the matrix of T with respect to $\mathcal{B}'$.

   (b) Let A be an $n \times n$ matrix over the complex numbers. **Prove**: A is similar to A^t.

Continued on the next page.

5. (8 pts) Let $R = M_n(F)$ where F is a field. For $1 \leq k \leq n$, let I_k be the subset of R consisting of matrices whose entries not on the k-th row are all zero.

   (a) Show that I_k is a right ideal of R.

   (b) Show that I_k is not a left ideal of R.

   (c) Give an example of a left ideal of R which is not a right ideal.

6. (12 pts)

   (a) Find the splitting field E of $x^4 - 2$ over $\mathbb{Q}$.

   (b) Find the Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$ by determining the order of the group G, finding a set of generators and their orders. Is G an Abelian group?