Algebra Qualifying Exam
January 5, 2010

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let G be a group of order $132 = 2^2 \cdot 3 \cdot 11$. Prove that G has a normal p-Sylow subgroup for some prime p that divides 132.

2. (a) Let S_1 and S_2 be commutative rings with 1. Consider the product ring $S_1 \times S_2$ with binary operations defined componentwise. Prove that (a, b) is a unit in $S_1 \times S_2$ if and only if a and b are units in S_1 and S_2 respectively.

   (b) Let p be a prime in $\mathbb{Z}$. Assume d_1 and d_2 are positive integers and $d_1 < d_2$. Consider the natural ring projection $\varphi : \mathbb{Z}/p^{d_2}\mathbb{Z} \to \mathbb{Z}/p^{d_1}\mathbb{Z}$; that is for any $\overline{m} \in \mathbb{Z}/p^{d_2}\mathbb{Z}$, $\varphi(\overline{m}) \equiv m \pmod{p^{d_1}}$. Prove that if $\overline{m} \in (\mathbb{Z}/p^{d_2}\mathbb{Z})^\times$, then $\varphi(\overline{m}) \in (\mathbb{Z}/p^{d_1}\mathbb{Z})^\times$. Thus φ induces a well-defined group homomorphism $\eta : (\mathbb{Z}/p^{d_2}\mathbb{Z})^\times \to (\mathbb{Z}/p^{d_1}\mathbb{Z})^\times$.

   (c) Prove that the induced homomorphism

   $$\eta : (\mathbb{Z}/p^{d_2}\mathbb{Z})^\times \to (\mathbb{Z}/p^{d_1}\mathbb{Z})^\times$$

   is surjective.

   (d) Prove that the natural surjective ring projection $\mathbb{Z}/500\mathbb{Z} \to \mathbb{Z}/25\mathbb{Z}$ induces a surjective homomorphism on the group of units

   $$(\mathbb{Z}/500\mathbb{Z})^\times \to (\mathbb{Z}/25\mathbb{Z})^\times.$$

   (*Hint*: Use the Chinese Remainder Theorem and the previous parts.)

3. Let G be a group, and let $Z(G)$ be the center of G.

   (a) Let $a \in G$. An *inner automorphism* of G is a function of the form $\gamma_a : G \to G$ given by $\gamma_a(g) = aga^{-1}$. Let $\mathrm{Inn}(G)$ be the set of all inner automorphisms of G. **Prove**: $\mathrm{Inn}(G) \cong G/Z(G)$.

   (b) Let ϕ be an automorphism of S_3. Show that ϕ permutes the set $\{(12), (13), (23)\}$, and no non-trivial automorphism of S_3 leaves all three elements of this set fixed. Deduce that all automorphisms of S_3 are inner automorphisms.

4. The splitting field E of $x^4 + 1$ over $\mathbb{Q}$ is a a simple extension. Find a primitive element for E and determine $\mathrm{Gal(E/\mathbb{Q})}$.

5. Suppose that $q = p^k$ for some positive integer k and some prime p. Let $\mathbb{F}_q$ denote the finite field with q elements.

   (a) Prove that $x^{p^k} - x$ is a seperable polynomial over $\mathbb{F}_p$. Prove that every element of $\mathbb{F}_q$ is a root of $x^{p^k} - x$.

   (b) What is the isomorphism type of $\mathbb{F}_q^\times$ as an abelian group? Explain.

   (c) Prove that the equation $a^3 = 1$ has 3 solutions in $\mathbb{F}_q$ if and only if $q \equiv 1 \pmod 3$.

6. Let p be an odd prime, and consider the group S_{2p}.

   (a) Let H be a p-Sylow subgroup of S_{2p}. Prove that H has order p^2, find the isomorphism type of H, and give generators for H.

   (b) How many p-Sylow subgroups does S_{2p} have?

7. Let R be a commutative ring with 1 and let I and J be ideals of R. Assume also that neither I nor J is the zero ideal and that neither I nor J contains 1. Let $\mathfrak{p}$ be a prime ideal in R containing IJ.

   (a) Prove that $\mathfrak{p}$ contains either I or J.

   (b) If I and J are comaximal (*i.e.* $I + J = R$), then IJ is *properly* contained in $\mathfrak{p}$.