Algebra Qualifying Exam
January 5, 2011

Do all 7 problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let G be a group of order pq where p and q are primes with $p < q$.
   (a) Prove that G is not a simple group.
   (b) Let P be a Sylow p-subgroup. Prove that G is cyclic if and only if P is normal in G.

2. (a) Prove that $\mathbb{F}_{11}[x]/(x^2+1)$ is a field consisting of 121 elements.
   (b) Construct a field of 81 elements.

3. Let R be a Euclidean Domain. Let m be the minimum integer in the set of norms of nonzero elements of R. Prove that every nonzero element of R of norm m is a unit. Deduce that a nonzero element of norm zero (if such an element exists) is a unit.

4. Suppose n is an integer with $n \geq 3$. Recall that the dihedral group D_{2n} is the group with the presentation

$$D_{2n} = \left\langle r, s \mid r^n = s^2 = 1, s^{-1}rs = r^{-1} \right\rangle.$$

   Prove that the center of D_{2n} is $\{1\}$ if n is odd and $\{1, r^{n/2}\}$ if n is even.

5. Let $f(x) = x^4 - 12x^2 + 25$.
   (a) Observe that $f(x+1) = x^4 + 4x^3 - 6x^2 - 20x + 14$. Using this or otherwise, prove that f is irreducible over $\mathbb{Q}$.
   (b) Determine the Galois group G of the splitting field of $f(x) = x^4 - 12x^2 + 25$ over $\mathbb{Q}$. Express each element of G as a permutation of the roots of f.

Continued on the other side.

6. Let R be a commutative ring with unity. We say that R is a *local ring* if R has a unique maximal ideal.

   (a) Prove that if R is a local ring with maximal ideal M, then every element of $R-M$ is a unit.

   (b) Conversely, prove that if the set of nonunits of R forms a proper ideal M; namely, $M \neq R$, then R is a local ring.

7. (a) **Prove**: If G is a group and $|G| = 2^2 \cdot 5 \cdot 19$, then G has a normal Sylow p-subgroup for either $p = 5$ or $p = 19$.

   (b) Let K be a Galois extension of $\mathbb{Q}$. **Prove**: If $[K : \mathbb{Q}] = 2^2 \cdot 5 \cdot 19$, then there exists some subfield E of K such that E is Galois over $\mathbb{Q}$ and $[K : E]$ is prime.