Algebra Qualifying Exam
January 6, 2012

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. In each of the following rings, determine if the designated set is a maximal ideal of the given ring. Justify your answers.
   (a) The set $\{3a + 2b \mid a, b \in \mathbb{Z}\}$ in the ring $\mathbb{Z}$.
   (b) The ideal $(x^3 + 4x + 2)$ in $\mathbb{Q}[x]$.
   (c) The ideal $(x - 2)$ in $\mathbb{Z}[x]$.
   (d) The ideal $(x - 2, x^2 + 2)$ in $\mathbb{Z}[x]$.
2. Let G be a group.
   (a) Let H be a normal subgroup of G. If H and G/H are Abelian, must G be Abelian? Either prove the statement or give a concrete example to show it is not true in general.
   (b) Let N be a normal subgroup of G and let H be any subgroup of G. Prove that NH is a subgroup of G. Give an example to show that NH need not be a subgroup of G if neither N nor H is normal.
3. Let K be the splitting field of $f(x) = x^4 - x^2 + 1$ over $\mathbb{Q}$. Find the Galois group of K over $\mathbb{Q}$. Express every element of $\mathrm{Gal}(K/Q)$ as a permutation of the roots of f.
4. Suppose that G is a group of order $595 = 5 \cdot 7 \cdot 17$.
   (a) **Prove**: G has a normal Sylow 5-subgroup.
   (b) **Prove**: G has a normal Sylow p-subgroup for either $p = 7$ or $p = 17$.
   (c) **Prove**: G has a normal subgroup of order 35 or a normal subgroup of order 85.
   (d) Recall that if H is a subgroup of a group G, then $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$. Using this or otherwise, prove that if a group of order 595 has a normal Sylow 7-subgroup, then it also has a normal Sylow 17-subgroup.

5. Let R be a commutative ring with 1, and let I be an ideal of R. We define the *Jacobson radical of* I (denoted $\operatorname{Jac} I$) to be the intersection of all maximal ideals of R that contain I. By convention, $\operatorname{Jac} R = R$.

   (a) Prove that $\operatorname{Jac} I$ is an ideal of R containing I.

   (b) We define the *radical of* I to be

   $$\operatorname{Rad} I = \{r \in R \mid r^m \in I \text{ for some } m \in \mathbb{Z}^+\}.$$

   Show that $\operatorname{Rad} I \subseteq \operatorname{Jac} I$.

   (c) Let $n > 1$ be an integer. Prove that $\operatorname{Jac} n\mathbb{Z} = m\mathbb{Z}$, where

   $$m = \operatorname{lcm}\{p_i \mid p_i \text{ is a prime dividing } n\}.$$

6. Let K be a finite extension of $\mathbb{Q}$. Let $f(x) \in \mathbb{Q}[x]$.

   (a) Suppose $[K : \mathbb{Q}] = 5$. Assume that all roots of f are in K, but not all of the roots are in $\mathbb{Q}$. Prove that K is the splitting field of f.

   (b) Give an explicit example to show that part (a) is false if $[K : \mathbb{Q}] = 4$.

   (c) Suppose K is a Galois extension of even degree over $\mathbb{Q}$. Assume that f is a separable and reducible polynomial of degree 4 such that all roots of f are in K, but none of them are in $\mathbb{Q}$. Prove that the Galois group of K over Q is not simple.

7. (a) Let P be a Sylow p-subgroup of H and let H be a subgroup of K. If $P \trianglelefteq H$ and $H \trianglelefteq K$, prove that P is normal in K.

   (b) Prove that if $P \in Syl_p(G)$ and $H = N_G(P)$, then $N_G(H) = H$. In other words, prove that *normalizers of Sylow p-subgroups are self-normalizing.*