Algebra Qualifying Exam
January 3, 2013

Do all seven problems. Each problem is worth 10 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Recall that the *center* of a ring R is the set

$$Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}.$$

   (a) Prove that the center of a division ring is a field.

   (b) Let R be a ring with identity $1 \neq 0$. Recall that $M_3(R)$ is the set of all 3×3 matrices with entries in R. Prove that if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ and } X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   are elements of $M_3(R)$ which commute, then $b = c = d = g = 0$.

   (c) Let R be a ring with identity $1 \neq 0$. Recall that a matrix $M = (a_{ij}) \in M_3(R)$ is called a *scalar matrix* if for some $a \in R$, $a_{ii} = a$ for $i = 1, 2, 3$ and $a_{ij} = 0$ for all $i \neq j$. Prove that the center of the matrix ring $M_3(R)$ is the set of scalar matrices.

2. Let K be the splitting field of $f(x) = x^4 - 12x^2 + 18$ over $\mathbb{Q}$. Find the Galois group of K over $\mathbb{Q}$.

3. Recall that a subgroup N of a group G is said to be *characteristic* if $f(N) \subseteq N$ for every automorphism f of G.

   (a) Prove that every characteristic subgroup is normal.

   (b) Let H and K be subgroups of G with $H \leq K$. Prove that if H is characteristic in K and K is a normal subgroup of G, then H is a normal subgroup of G.

4. Assume that G is a group of order $782 = 2 \cdot 17 \cdot 23$. Let P be a Sylow-17 subgroup of G, and let Q be a Sylow-23 subgroup of G.

   (a) Prove that Q is normal in G.

   (b) Prove that PQ is a normal subgroup of G and that P is a characteristic subgroup of PQ.

   (c) Prove that P is a normal subgroup of G.

5. Let F be a field, and let α be an element of some extension field K of F.

   (a) Prove that if $[F(\alpha) : F]$ is a power of 2, then $F(\alpha^3) = F(\alpha)$.

   (b) **True or False:** If $[F(\alpha) : F] = 6$, then $F(\alpha^3) = F(\alpha)$. Justify your answer with a proof or a counterexample.

6. A group G is said to be *metabelian* if it has a subgroup N such that N is Abelian, N is normal in G and G/N is Abelian.

   (a) Give an example of a finite group that is metabelian but not Abelian.

   (b) Prove that every homomorphic image of a metabelian group is metabelian.

7. Let R be a commutative ring and $a, b \in R$ such that $b \neq 0$. Recall that a non-zero element $d \in R$ is called *a greatest common divisor* of a and b if:

   - $d|a$ and $d|b$, and
   - if $d'|a$ and $d'|b$ then $d'|d$.

   (a) Let $R = \mathbb{Z}[\sqrt{-5}]$. Fix $a := 6$ and $b := 2+2\sqrt{-5}$ and let N denote the usual norm on $\mathbb{Z}[\sqrt{-5}]$ defined by $N(x + y\sqrt{-5}) = x^2 + 5y^2$. Using this norm, prove that a and b do not have a greatest common divisor.

   (b) Prove that $R = \mathbb{Z}[\sqrt{-5}]$ is not a Principal Ideal Domain.