Algebra Qualifying Exam
January 7, 2014

Do all seven problems. Each problem is worth 10 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let k be a positive integer. **Prove**: No group of order $2^2 \cdot 5 \cdot 19^k$ is simple. (**Note:** If you use the Classification of Finite Simple Groups, you should prove it.)

2. Let $\mathbb{F}_3$ be the finite field with 3 elements.

   (a) Show that the quotient ring $K = \mathbb{F}_3[x]/(x^2+1)$ is a finite field with 9 elements.

   (b) Show that the map $\phi : K \to K$, $\phi(z) = z^3$ is an automorphism of K, and find all other automorphisms of the field K.

   (c) By explicitly giving a generator, show that $K^\times$ is a cyclic group.

3. Let B be a commutative ring and A a subring of B (so that $1 \in A$). An element x of B is said to be *integral* over A if x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

   where the a_i are elements of A. You may assume that the set C of elements of B which are integral over A is a subring of B containing A. We say that B is *integral* over A if $C = B$. Suppose that B is integral over A.

   (a) Let $\mathfrak{b}$ be an ideal of B and $\mathfrak{a} = A \cap \mathfrak{b}$. Prove that $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.

   (b) Prove that the field of fractions of B is integral over the field of fractions of A.

   (c) Suppose further that $A \subseteq B$ are integral domains. Prove that B is a field if and only if A is a field.

   (d) Let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q} \cap A$. Prove that $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is maximal.

4. Let k be a positive integer.

   (a) Prove that if $k \geq 2$, then $(2^{k-1}+1)^2 \equiv 1 \pmod{2^k}$.

   (b) Determine, with proof, the set of all positive integers k such that $(\mathbb{Z}/2^k\mathbb{Z})^\times$ is cyclic.

5. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible cubic polynomial, and let K be a splitting field of $f(x)$ over $\mathbb{Q}$. Suppose that the Galois group $\text{Aut}(K/\mathbb{Q})$ is of order 3.

   (a) Show that K is isomorphic to $\mathbb{Q}[x]/(f(x))$.

   (b) Show that K is isomorphic to a subfield of the field of real numbers.

6. Let G be a group. Recall that if $h, k \in G$, then the *commutator* of h and k is defined as $[h,k] = hkh^{-1}k^{-1}$. The commutator subgroup of G is

$$G' = \langle [h,k] : h,k \in G \rangle.$$

   (a) **Prove**: G' is a characteristic subgroup of G.

   (b) **Prove**: If $n \geq 5$ and $G = S_n$, then $G' = A_n$.

7. Prove that $\mathbb{Q}[x,y]/(y^2 - x^3)$ is isomorphic to the ring $\mathbb{Q}[t^2, t^3]$.