Algebra Qualifying Examination

January 7, 2015

Do all seven (7) problems. Each problem is 10 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

Notation. Throughout the test, we use the following notation:
$\mathbf{Z}$ is the ring of integers.
$\mathbf{Q}$ is the field of rational numbers.
S_n is the permutation group on n elements.

(1) Let G be a group. The *commutator subgroup* of G is the group

$$G' = \langle xyx^{-1}y^{-1} : x, y \in G \rangle.$$

Prove each of the following statements.

(a) The set G' is a normal subgroup of G.

(b) If N is a normal subgroup of G and G/N is abelian, then G' is a subgroup of N.

(c) If H is a subgroup of G and $G' \subseteq H$, then H is normal in G.

(d) If $n \geq 3$. then the commutator subgroup of S_n is its alternating group A_n.

(2) Let $k \geq 1$ be an integer. Prove that if G is a group of order $2^2 \cdot 3 \cdot 11^k$, then G is not simple.

(3) Give examples of each of the following. Be sure you justify your assertions.

(a) A ring R and an ideal $\mathfrak{a}$ in R which is not principal.

(b) A ring S and a nonzero ideal $\mathfrak{p}$ in S which is prime but not maximal.

(c) An integral domain which is not a UFD, and give an explicit example of nonunique factorization.

(d) A polynomial in $\mathbf{Q}[X]$ which is irreducible but not Eisenstein at any prime $p \in \mathbf{Z}$.

(4) (a) Define a Euclidean domain.

(b) Show that $F[X]$ is Euclidean, where F is a field. (If you wish to cite the Euclidean Algorithm Theorem, then you must prove it.)

(5) (a) Decompose the polynomial $f(X) = X^4 - 9X^2 + 14$ into a product of irreducibles in $\mathbf{Q}[X]$. Be sure to prove that your irreducible factors are indeed irreducible.

(b) Compute the splitting field K of $f(X)$ over $\mathbf{Q}$.

(c) Compute $\mathrm{Gal}(K/\mathbf{Q})$.

(d) Use Galois theory to prove that the polynomial $X^4 - 18X^2 + 25$ is irreducible.

(6) (a) Let $\mathbf{F}_2$ be the field with 2 elements. Define the field of rational functions in the indeterminate T over $\mathbf{F}_2$ by

$$L = \mathbf{F}_2(T) = \left\{ \frac{f(T)}{g(T)} : f(T), g(T) \in \mathbf{F}_2[T],\ g(a) \neq 0 \text{ for some } a \in \mathbf{F}_2 \right\}.$$

Prove that $L(\sqrt{T})$ is a finite extension of L of degree 2 but is not a Galois extension.

(b) State and prove necessary and sufficient conditions for $\mathbf{Q}(\sqrt{d_1})$ and $\mathbf{Q}(\sqrt{d_2})$ to be isomorphic as field extensions of $\mathbf{Q}$, where d_1 and d_2 are squarefree in $\mathbf{Q}$.

(7) Let $K = \mathbf{Q}(\sqrt[3]{2} + \omega)$ be a field extension of $\mathbf{Q}$ where ω is a root of $X^2 + X + 1$.

(a) Prove that $\sqrt[3]{2} \in K$. Deduce that $K = \mathbf{Q}(\sqrt[3]{2}, \omega)$.

(b) Prove that $\mathrm{Gal}(K/\mathbf{Q}) \cong S_3$.

(c) Display the lattice of subgroups of S_3. For each subgroup, find a set of corresponding generators in $\mathrm{Gal}(K/\mathbf{Q})$.

(d) For each nontrivial proper subgroup of $\mathrm{Gal}(K/\mathbf{Q})$, find its fixed field.