Algebra Qualifying Exam
January 6, 2017

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Consider S_4, the permutation group on 4 symbols.
   (a) **Prove**: S_4 does not have a normal subgroup of order 3.
   (b) Assume that H is a normal subgroup of S_4 of order 8.
   Prove: $\{\sigma \in S_4 : \sigma^{-1} = \sigma\} \subseteq H$.
   (c) **Prove**: S_4 does not have a normal subgroup of order 8.

2. Suppose that r is a positive integer and $2^r - 1$ is prime.
   Prove: There are no simple groups of order $2^r(2^r - 1)$.

3. (a) Let R be an integral domain and let P be a prime ideal of R. Let

   $$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[x],$$

   where $n \geq 1$. Assume that $a_0, a_1, \ldots, a_{n-1} \in P$ and $a_0 \notin P^2$. **Prove**: $f(x)$ is irreducible in $R[x]$.
   (b) **Prove**: The polynomial in two variables $x^2 + y^2 - 1$ is irreducible over $\mathbb{Q}$.

4. Let $f(x) = x^4 + ax^2 + b \in \mathbb{Z}[x]$ be irreducible. Let G be the Galois group of $f(x)$ Note that f is an even function, so we may write the roots of f as $\alpha, \beta, -\alpha, -\beta$.
   Prove: If $\alpha\beta \in \mathbb{Q}$, then G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

5. Let $\zeta = e^{2\pi i/5}$.
   (a) **Prove**: There exists a unique quadratic subfield of $L \subset \mathbb{Q}(\zeta)$. ($L$ is quadratic if and only if $[L : \mathbb{Q}] = 2$.)
   (b) Let L be as in part (a). **Prove**: $L = \mathbb{Q}(\zeta + \zeta^{-1})$.
   (c) Let $\alpha = \zeta + \zeta^{-1}$. **Prove**: $\alpha^2 + \alpha - 1 = 0$. Deduce that $L = \mathbb{Q}(\sqrt{5})$.

6. (a) Give an example of a principal ideal domain R such that $R[x]$ is not a principal ideal domain. Justify your answer.

   (b) Let R be a commutative ring with 1 and let N be a free R-submodule of R. **Prove:** Rank $N \leq 1$.

   (c) Give an example of a free module M and its submodule N over a commutative ring R such that N is not free. Justify your answer.

7. Let M be a module over a ring R. Recall that the annihilator of M over R is the ideal

$$\text{Ann}_R(M) = \{r \in R : rM = 0\}.$$

   (a) Suppose that M is a module over a principal ideal domain R and $\text{Ann}_R(M) = (c)$ for some $c \in R$ such that $c = ab$, where $a, b \in R$ and $\gcd(a, b) = 1$.
   Prove: $M = M_a \oplus M_b$. where

$$M_a = \{m \in M : am = 0\} \text{ and } M_b = \{m \in M : bm = 0\}.$$

   (b) Let V be a finite dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation with the minimal polynomial $f(x) \in F[x]$.
   Prove: $\text{Ann}_{F[x]}(V) = (f(x))$.

   (c) Let V be a finite dimensional vector space over $\mathbb{Q}$, and let $T : V \to V$ be a linear transformation with the minimal polynomial $f(x) = x^4 - x^3 - x + 1$.
   Prove: $V = U \oplus W$ where

$$U = \{v \in V : T^2(v) = 2\,T(v) - v\} \text{ and } W = \{v \in V : T^2(v) = -T(v) - v\}.$$