Algebra Qualifying Exam (Version I)
August 23, 2017 (1pm–5pm)

This version is for those students who have completed their latest MTH 625 course Spring 2016 or after. This version is also for those students who have not completed MTH 623 or/and MTH 625.

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. No calculators or other electronic devices are allowed.
4. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
5. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.
6. All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.
7. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
8. In order to pass the algebra qualifying exam, you must receive a passing score 70% or better. Equivalently, the passing grade for the algebra qualifying exam is 49 out of 70.

Notation. Throughout the exam, we use the following notation:

$\mathbb{Z}$ is the ring of integers;

$\mathbb{Q}$ is the field of rational numbers;

$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n, where n is a positive integer;

$F^\times$ is the set of nonzero elements of a field F.

(1) Recall that the commutator subgroup of a group G, denoted $[G, G]$, is the subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$ for all $x, y \in G$. Let $p > 2$ be a prime and set

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}/p\mathbb{Z},\ ad \in (\mathbb{Z}/p\mathbb{Z})^\times \right\}.$$

- (a) (3 points) Show that $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{Z}/p\mathbb{Z}\}$ is a cyclic group of order p.
- (b) (3 points) Show that $N = [B, B]$.
- (c) (4 points) Show that $B/N \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times$.

(2) (10 points) Let G be a group of order $1645 = 5 \cdot 7 \cdot 47$. Prove that G has unique $5, 7, 47$ Sylow subgroups and that G is abelian.

(3) (10 points) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ be a polynomial of degree n and let p a prime in $\mathbb{Z}$ such that for some $k < n$,

- (a) p divides a_i for all $i = 0, 1, \ldots, k-1$,
- (b) p does not divide $a_k a_n$,
- (c) p^2 does not divide a_0.

Show that $f(x)$ has at least one irreducible factor of degree $\geq k$.

(4) We say that a ring homomorphism $f \colon R \to S$ is *unital* if $f(1_R) = 1_S$ where 1_R and 1_S denote the multiplicative identities in rings R and S respectively.

Find all unital ring hommorphisms:

- (a) (2.5 points) $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/35\mathbb{Z}$;
- (b) (2.5 points) $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}$;
- (c) (2.5 points) $\mathbb{Z} \to \mathbb{Z}$;
- (d) (2.5 points) $\mathbb{Q} \to \mathbb{Z}$.

Justify your answers.

(5) Let K/F be an extension of fields and let R be a ring such that $F \subseteq R \subseteq K$.

(a) (7 points) If K/F is an algebraic extension, prove that R is a field.

(b) (3 points) Give an example where K/F is an infinite extension and R is not a field. Justify your answer.

(6) (a) (6 points) Give an example of a Galois extension of fields of degree 6. Compute its Galois group. Justify your answers.

(b) (4 points) Give an example of a field extension of degree 6 that is not Galois. Justify your answer.

(7) (a) (4 points) Let R be a commutative ring with 1. For a short exact sequence

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$

of R-modules, if there exists an R-module homomorphism $f\colon C \to B$ such that $\phi \circ f$ is the identity map on C, prove that $A \oplus C \cong B$ (as R-modules) by explicitly constructing an isomorphism. In this case we say that the short exact sequence is *split*.

(b) (3 points) Give an example of a short exact sequence of $\mathbb{Z}$-modules that is not split. Justify your answer.

(c) (3 points) If M is a finitely generated torsion free $\mathbb{Z}$-module, prove that every short exact sequence $0 \longrightarrow L \longrightarrow N \longrightarrow M \longrightarrow 0$ of $\mathbb{Z}$-modules is split. In this case we say that the $\mathbb{Z}$-module M is *projective*. (If you use the structure theorem or classification of finitely generated modules over a PID, you need to prove it.)

(8) (i) Let R be a commutative ring.

(a) (2 points) Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. Prove $(R/\mathfrak{a}) \otimes_R (R/\mathfrak{b}) \cong R/(\mathfrak{a}+\mathfrak{b})$.

(b) (3 points) Let $\mathfrak{a} \subset R$ be an ideal. Let M and N be R-modules such that $\mathfrak{a}N = 0$, so N is naturally an $R/\mathfrak{a}$-module by $\overline{r}n = rn$ for $\overline{r} \in R/\mathfrak{a}$ and $n \in N$. Show

$$M \otimes_R N \cong M/\mathfrak{a}M \otimes_{R/\mathfrak{a}} N$$

as $R/\mathfrak{a}$-modules, where $M \otimes_R N$ is given the structure of an $R/\mathfrak{a}$-module from N being an $R/\mathfrak{a}$-module ($\overline{r}(m \otimes n) = m \otimes rn$).

(ii) (a) (3 points) Give an example M of an infinite $\mathbb{Z}$-module such that $M \otimes_{\mathbb{Z}} M = 0$. Justify your answer.

(b) (2 points) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules.