

Algebra Qualifying Exam (Version II)
August 23, 2017 (1pm–5pm)

This version is for those students who have completed their **latest MTH 625** course before Spring 2016.

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. No calculators or other electronic devices are allowed.
4. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
5. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.
6. All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.
7. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
8. In order to pass the algebra qualifying exam, you must receive a passing score 70% or better. Equivalently, the passing grade for the algebra qualifying exam is 49 out of 70.

Notation. Throughout the exam, we use the following notation:

\mathbb{Z} is the ring of integers;

\mathbb{Q} is the field of rational numbers;

$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n , where n is a positive integer;

\mathbb{F}_q is the field with q elements, where $q = p^k$ for some prime p and $k \geq 1$;

F^\times is the set of nonzero elements of a field F .

- (1) Recall that the commutator subgroup of a group G , denoted $[G, G]$, is the subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$ for all $x, y \in G$. Let $p > 2$ be a prime and set

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}/p\mathbb{Z}, ad \in (\mathbb{Z}/p\mathbb{Z})^\times \right\}.$$

- (a) (3 points) Show that $N = \{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}/p\mathbb{Z} \}$ is a cyclic group of order p .
- (b) (3 points) Show that $N = [B, B]$.
- (c) (4 points) Show that $B/N \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times$.
- (2) (10 points) Let G be a group of order $1645 = 5 \cdot 7 \cdot 47$. Prove that G has unique 5, 7, 47 Sylow subgroups and that G is abelian.

- (3) Let $p > 2$ be a prime.

- (a) (4 points) Prove that the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.
- (b) (3 points) Let H be a subgroup of a group G such that $[G : H] = 2$ and let $a, b \in G$. If $a, b \notin H$, show that $ab \in H$.
- (c) (3 points) Suppose that m and n are distinct integers. Show that at least one of m , n or mn is a square modulo p . Here we say that an integer a is a square modulo p if there exists an integer b such that $a \equiv b^2 \pmod{p}$.
- (4) (10 points) Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ be a polynomial of degree n and let p a prime in \mathbb{Z} such that for some $k < n$,
- (a) p divides a_i for all $i = 0, 1, \dots, k - 1$,
- (b) p does not divide $a_k a_n$,
- (c) p^2 does not divide a_0 .

Show that $f(x)$ has at least one irreducible factor of degree $\geq k$.

- (5) We say that a ring homomorphism $f: R \rightarrow S$ is *unital* if $f(1_R) = 1_S$ where 1_R and 1_S denote the multiplicative identities in rings R and S respectively.

Find all unital ring homomorphisms:

- (a) (2.5 points) $\mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/35\mathbb{Z}$;
- (b) (2.5 points) $\mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}$;
- (c) (2.5 points) $\mathbb{Z} \rightarrow \mathbb{Z}$;
- (d) (2.5 points) $\mathbb{Q} \rightarrow \mathbb{Z}$.

Justify your answers.

- (6) Let K/F be an extension of fields and let R be a ring such that $F \subseteq R \subseteq K$.

- (a) (7 points) If K/F is an algebraic extension, prove that R is a field.
- (b) (3 points) Give an example where K/F is an infinite extension and R is not a field. Justify your answer.

- (7) (a) (6 points) Give an example of a Galois extension of fields of degree 6. Compute its Galois group. Justify your answers.
- (b) (4 points) Give an example of a field extension of degree 6 that is not Galois. Justify your answer.

- (8) (a) (3 points) Describe the splitting field F for the polynomial $x^{16} - 1 \in \mathbb{Q}[x]$. Find the degree $[F : \mathbb{Q}]$.
- (b) (4 points) Find the splitting field for the polynomial $x^4 + 1 \in (\mathbb{Z}/3\mathbb{Z})[x]$.
- (c) (3 points) Prove that $\mathbb{F}_{125}/\mathbb{F}_5$ is a Galois extension. Describe the Galois group of this extension by explicitly writing down the automorphisms.

Justify all your answers.