Algebra Qualifying Exam
August 24, 2019 (1pm–5pm)

For *page arrangement*, follow these general instructions closely throughout the test.

- Write in a legible fashion.
- For each sheet, write only on a single side and leave a 1×1 square inch area blank at the upper left corner – for staples.
- Start with a new sheet for each question.
- Absolutely no cell phones of any kind during the entire test.
- Arrange all papers by the order of the questions before submission.

The instructions 1-8 are as given in ″Algebra Qualifying Exam August 2019 Guidelines″ under ″Exam Format″. Instruction #9 is additional. It is the student's responsibility to read and follow these instructions carefully.

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. No calculators or other electronic devices are allowed.
4. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
5. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.
6. All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.
7. If multiple version of solutions are provided, then all versions will be graded and then the average credits is taken as the final score.
8. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
9. **Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.**
10. Exam grade: In order to pass the algebra qualifying exam, you must receive a passing score 70% or better.

Notation. Throughout the exam, we use the following notation:

$\mathbb{Z}$ is the ring of integers;
$\mathbb{Q}$ is the field of rational numbers;
$\mathbb{R}$ is the field of real numbers;
$\mathbb{C}$ is the field of complex numbers.
$n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$, where n is a positive integer;
$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n, where n is a positive integer.

(1) Let G be a group. For any two elements g_1 and g_2, we say that g_1 and g_2 are *conjugate* in G (or equivalently g_1 is conjugate in G to g_2), if there exists an element $h \in G$ such that $g_1 = h^{-1}g_2h$.

(a) (5 points) Consider S_n, the group of permutations of n elements. Prove that for any element $g \in S_n$, g is conjugate in S_n to g^{-1}.

(b) (5 points) Consider the alternative group A_n, a group generated by even permutations. Prove that there exists an element in A_4 that is NOT conjugate in A_4 to its inverse.

(2) For a finite group G and a prime p, let $\mathrm{Syl}_p(G)$ denote the set of all p-Sylow subgroups.

(a) (3 points) Let H be a finite group. Let p and q be two distinct primes dividing $|H|$. Suppose $\mathrm{Syl}_p(H) = \{P\}$ and $\mathrm{Syl}_q(H) = \{Q\}$. Prove that $xy = yx$ for all $x \in P$ and $y \in Q$.

(b) (2 points) Let G be a finite group and for $g \in G$, we denote the order of g by $o(g)$. Assume that $x, y \in G$ are such that $xy = yx$ and $\gcd(o(x), o(y)) = 1$. Prove that $o(xy) = o(x)o(y)$.

(c) (3 points) Let G be a finite group and $|G| = p_1^{n_1} \cdots p_m^{n_m}$ where p_i is a prime and $n_i \geq 1$ is an integer for all i, $1 \leq i \leq m$, and $p_i \neq p_j$ if $i \neq j$. Assume that $\mathrm{Syl}_{p_i}(G) = \{P_i\}$ for $1 \leq i \leq m$.

Defne a map $\phi : P_1 \times \cdots \times P_m \to G$ as follows:

$$\phi(x_1, \ldots, x_m) = x_1 \cdots x_m, \text{ for all } x_i \in P_i.$$

Here $P_1 \times \cdots \times P_m$ denotes the direct product of $P_1, \ldots, P_m$. Prove that ϕ is an isomorphism of groups.

(d) (2 points) Suppose that G is a finite abelian group with a property that all its Sylow subgroups are cyclic. Prove that G is cyclic.

(3) Let R and S be commutative rings with 1. Let $h : R \to S$ be a ring homomorphism. For any subset B of S, we define the *inverse mage* of a subset B to be the set

$$h^{-1}(B) = \{r \in R : h(r) \in B\}$$

(a) (3 points) Let J be an ideal in S. Prove that the $h^{-1}(J)$ is an ideal in R.

(b) (4 points) Assume that $\mathfrak{q}$ is a prime ideal in S. Prove that $h^{-1}(\mathfrak{q})$ is a prime ideal of R.

(c) (3 points) Assume I is an ideal in R. Define the set $h(I)$ to be

$$h(I) = \{b \in S : b = h(a) \text{ for some } a \in I\}$$

Is $h(I)$ always an ideal in S? *Prove the statement if yes. Otherwise, provide concrete examples for R, S, h and I and* elaborate *why they work as a counterexample.*

(4) Let $A = k[x]$ be a polynomial ring over a field k. This question involves Chinese Remainder Theorem that you may use without a proof.

(a) (3 poins) Applying the long division algorithm on a polynomial ring, prove that every ideal in A is generated by one element. (This is also how the statement "*a Euclidean domain is a principal ideal domain*" is proved. However directly applying this statement without a proof will obtain only few to no credit.)

(b) (3 points) Assume $A = \mathbb{Q}[x]$. Let I be the ideal in A generated by

$$\begin{aligned} &x^8 + x^6 - x^2 - 1, \\ &x^6 - x^5 + x^4 - x^2 + x - 1, \\ &x^6 + 2x^4 + 2x^2 + 1. \end{aligned}$$

By Part (a), I is a principal ideal. Find a polynomial $f(x)$ such that I is generated by $f(x)$.

(c) (1 point) Let A and I be as in Part (b). Decompose A/I into a product of integral domains. Otherwise prove that A/I is already an integral domain.

(d) (1 point) Let J be the principal ideal generated by $(x+1)^3(x-1)$. Consider A/J as in Part (c). Write A/J into a product of integral domains.

(e) (2 points) Assume $A = \mathbb{Q}[x]$. Does there exist a polynomial $g(x)$ in A such that $g(x)$ is a multiple of $(x+1)^2$ and that $g(x)$ has a remainder 1 when divided by $x^2 - 1$? Must fully justify your answer.

(5) Let K be a degree 5 extension of $\mathbb{Q}$ and let $f(x)$ be a polynomial in $\mathbb{Q}[x]$, not necessarily irreducible. Assume that $f(x)$ has all its roots in K but not all of them are in $\mathbb{Q}$.

(a) (5 points) Let E be the splitting field of $f(x)$. Prove that $E = K$.

(b) (5 points) Show by precise examples for K and $f(x)$ that the statement in Part (a) does not necessarily hold if $[K : \mathbb{Q}] = 6$.

(6) Let L be the field of rational functions in one variable over $\mathbb{C}$; *i.e.*

$$L = \mathbb{C}(y) = \{\frac{f(y)}{g(y)} | f(y), g(y) \in \mathbb{C}[y]\}$$

Let $M = \mathbb{C}(\beta)$ with $\beta = y^5$ and let $N = \mathbb{C}(\alpha)$ with $\alpha = -\dfrac{(y^5-1)^2}{y^5} \in M$. Notice that now we have

$$\mathbb{C} \subseteq N = \mathbb{C}(\alpha) \subseteq M = \mathbb{C}(\beta) \subseteq L = \mathbb{C}(y)$$

(a) (3 points) Prove that M is an extension of N of degree 2.

(b) (4 points) Prove that L is an extension field of M of degree 5. (*Hint*: Utilize all $5th$ roots of unity contained in $\mathbb{C}$).

(c) (3 points) Prove that L is a splitting field of some polynomial of over N. And deduce that L is a Galois extension of N of degree 10.

(7) Let $V = \mathbb{R}^n$ be a two-dimensional vector space over $\mathbb{R}$ and let $S = \mathbb{R}[t]$ be a polynomial ring of one variable. For a given $n \times n$ matrix E, we may consider V as a left S-module via E by left multiplication

$$t \cdot \mathbf{v} = E\mathbf{v}$$

for any vector $\mathbf{v} \in V$.

(a) (3 points) Prove that if W is an S-submodule of V, then W is a vector subspace of V that is invariant under E; namely, $E(W) \subseteq W$.

(b) (3 points) Take $n = 2$. Assume that $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Prove that V is an irreducible S-module. (Recall that a module is irreducible if it has no nontrivial proper submodule; namely, the only submodules are 0 and itself.)

(c) (4 points) Take $n = 3$. Assume that $E = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find submodules of V that have dimension 1 and dimension 2 respectively.

(8) (a) (5 points) Let R denote an integral domain. Prove that an ideal I of R is a free R-module if and only if I is principal.

(b) (5 points) Recall: We say that an R-module M is torsion free if for each $0 \neq m \in M$, we have $\{r \in R \ : \ rm = 0\} = \{0\}$.

Give an example of an integral domain R and a finitely generated torsion free R-module M that is not a free R-module. Justify your answer.