Algebra Qualifying Exam
January 10, 2020 (1pm–5pm)

For *page arrangement*, follow these general instructions closely throughout the test.

- Write in a legible fashion.
- For each sheet, write only on a single side and leave a 1×1 square inch area blank at the upper left corner – for staples.
- Start with a new sheet for each question.
- Arrange all papers by the order of the questions before submission.

The instructions 1-9 are as given in ″Algebra Qualifying Exam January 2020 Guidelines″ under ″Exam Format″. Instruction #10 is additional. It is the student's responsibility to read and follow these instructions carefully.

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. No calculators or other electronic devices are allowed.
4. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
5. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.
6. All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.
7. If multiple version of solutions are provided, then all versions will be graded and then the average credits is taken as the final score.
8. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
9. In order to pass the algebra qualifying exam, you must receive a total score of at least 49 (out of 70) **AND** your solutions to at least 3 questions (out of 7) should receive full credit.
10. **Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.**

Notation. Throughout the exam, we use the following notation:

$\mathbb{Z}$ is the ring of integers;
$\mathbb{Q}$ is the field of rational numbers;
$\mathbb{C}$ is the field of complex numbers;
$n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$, where n is a positive integer;
$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n, where n is a positive integer.

(1)
   (a) (4 points) Let $H = \langle h \rangle$ be a cyclic group, generated by an element h, of order n. Assume that m is a number relatively prime to n. Prove that $H = \langle h^m \rangle$.
   (b) (3 points) Assume that G is a finite group of odd order. Prove that every element of G is a square; that is, for any $h \in G, h = g^2$ for some $g \in G$.
   (c) (3 points) Prove a generalization of the result in Part (b): if G is a group of order n and k is an integer relatively prime to n, then a function $\phi : G \to G$ given by $\phi(g) = g^k$ is bijective. Note that ϕ is not necessarily a group homomorphism.

(2) Let G denote a simple group of order 60 with a number of 2-Sylow subgroups, $n_2 \leq 10$. By completing the following 3 parts, you will prove that G is isomorphic to A_5, the alternating group of degree 5.
   (a) (4 points) Prove that $n_2 = 5$.
   (b) (3 points) Prove that G is isomorphic to a subgroup H of S_5.
   (c) (3 points) Prove that $H = A_5$ where H is obtained from Part (b).

(3) Consider a subring R of $\mathbb{C}$ defined by

$$R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \ : \ a, b \in \mathbb{Z}\}.$$

   (a) (3 points) Find an ideal I of a polynomial ring $\mathbb{Z}[x]$ such that R is isomorphic to the quotient ring $\mathbb{Z}[x]/I$. Justify your answer.
   (b) (1 point) Define a norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$. Prove that N is multiplicative, i.e., $N(zw) = N(z)N(w)$ for $z, w \in R$.
   (c) (1 point) Prove that R is an integral domain.
   (d) (1 point) Prove that the units in R are 1 and -1.
   (e) (2 points) Prove that $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are all irreducible elements.
   (f) (2 points) Prove that R is not a unique factorization domain (UFD).

(4) Let R denote a subring of $\mathbb{Q}$ containing 1. We denote the absolute value of an integer n by $|n|$.

   (a) (4 points) Assume that $m, n \in \mathbb{Z}$ are non-zero relatively prime integers with $\frac{m}{n} \in R$. Prove that $\frac{1}{n} \in R$.

   (b) (6 points) Let I denote a non-zero ideal of R. Let $0 \neq b, a \in \mathbb{Z}$ be such that $\frac{a}{b} \in I$ and $|a| = \min\{|c| : 0 \neq \frac{c}{d} \in I\}$. Prove that I is a principal ideal generated by $\frac{a}{b}$.

(5) (a) (4 points) Let $E_1 = \mathbb{Q}(\sqrt{2+\sqrt{2}})$ be a field extension of $\mathbb{Q}$. Prove that E_1 is a Galois extension of $\mathbb{Q}$ with cyclic Galois group.

   (b) (3 points) Let $E_2 = \mathbb{Q}(\sqrt{3+\sqrt{5}})$ be a field extension of $\mathbb{Q}$. Prove that E_2 is a Galois extension of $\mathbb{Q}$ with its Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   (c) (3 points) Let $E_3 = \mathbb{Q}(\sqrt{1+\sqrt{2}})$ be a filed extension of $\mathbb{Q}$. Prove that E_3 is not a Galois extension of $\mathbb{Q}$. Determine the degree of extension $[E_3 : \mathbb{Q}]$ and the group $\mathrm{Aut}(E_3/\mathbb{Q})$ of field automorphisms of E_3 fixing $\mathbb{Q}$.

(6) Let $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathbb{Q}[x]$ with following conditions:

$$a_4 \neq 0, a_4 = a_0, a_3 = a_1.$$

This type of a polynomial is known as a *palindromic polynomial*. Let K denote the splitting field of $p(x)$.

   (a) (3 points) Prove that there exists a degree 2 polynomial $q(x) \in \mathbb{Q}[x]$ such that

$$x^{-2}p(x) = q\left(x + \frac{1}{x}\right).$$

   (b) (1 point) Let $q(x)$ be the polynomial as in (a). Suppose $q(\beta) = 0$. Let $L = \mathbb{Q}(\beta)$. Prove that $[L : \mathbb{Q}] \leq 2$.

   (c) (3 points) Prove that if $p(r) = 0$, then $r \neq 0$ and $p(r^{-1}) = 0$.

   (d) (3 points) Prove that $[K : \mathbb{Q}] \leq 8$.

(7) Let $M_3(\mathbb{C})$ denote a set of all 3×3 matrices over $\mathbb{C}$. For $A \in M_3(\mathbb{C})$, let $c_A(x)$ denote the characteristic polynomial of A and $m_A(x)$ denote the minimal polynomial of A.

   (a) (5 points) Determine all possible pairs $(c_A(x), m_A(x))$ for $A \in M_3(\mathbb{C})$ with only eigenvalues -1 and 2.

   (b) (5 points) For each pair obtained in Part (a), give an explicit example of $A \in M_3(\mathbb{C})$ associated with that pair, i.e. with prescribed characteristic polynomial and minimal polynomial.

(8) (a) (2 points) Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Justify your answer.

(b) (4 points) Suppose that $(A, +)$ is a *divisible* abelian group, i.e. for each $a \in A$ and each non-zero integer n, there exists $x \in A$ such that $nx = a$. Let $(B, +)$ be an abelian group in which every element has finite order. Note that A and B are $\mathbb{Z}$-modules. Prove that $A \otimes_{\mathbb{Z}} B = 0$

(c) (4 points) Let $K = F(\alpha)$ denote a finite simple field extension over F with degree of extension $[K : F] \geq 2$. Prove that $K \otimes_F K$ is not an integral domain.