Algebra Qualifying Exam
January 4, 2018

Read the following instructions very carefully.

By proceeding to write the exam, it implies that you understand the following instructions and agree that your exam will be graded according to these statements.

Instructions:

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
4. **You may use all theorems and properties freely but you must explain precisely how they are applied to reach your claim.**
5. All your submitted written responses for the first seven problems will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase or indicate clearly the responses you don't want to be graded.
6. If multiple versions of solutions are provided, then all versions will be graded and then the average credits are taken as the final score.
7. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
8. No calculators or other electronic devices are allowed.
9. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.

Exam grade: In order to pass the algebra qualifying exam, you must receive a passing score 70% or better. Equivalently, the passing grade for the algebra qualifying exam is 49 out of 70.

1. Let p be a prime dividing the order of a finite group G.

   (a) (6 pts) Show that if H is a normal subgroup of G, and if the prime p does not divide the index $[G : H]$, then H contains every Sylow p-subgroup of G.

   (b) (4 pts) Give a counterexample when H is not normal. Justify your answer.

2. (a) (5 pts) Let H be a subgroup of a group G. Let $S = G/H$ denote the set of all left cosets of H in G. Consider the action of G on S by left multiplication:

$$g \cdot aH = gaH \quad \text{for all } g, a \in G.$$

   Prove that $\bigcup_{g \in G} gHg^{-1} = \bigcup_{s \in S} \mathrm{stab}_G(s)$ where $\mathrm{stab}_G(s)$ is the stabilizer of s in G.

   (b) (5 pts) Prove that for a proper subgroup H of a finite group G, $\bigcup_{g \in G} gHg^{-1} \neq G$.

3. **True or False.** *If the statement is true, you need to prove it. If the statement is false, you need to provide a concrete counterexample and justify.*
   Let R and S be nonzero commutative rings with identities.

   (a) (5 pts) Any ideal K in the direct product ring $R \times S$ is of the form $I \times J$ where I is an ideal in R and J is an ideal in S.

   (b) (5 pts) If $\phi\colon R \to S$ is a ring homomorphism, then the inverse image of any nonzero maximal ideal in S (under ϕ) is a maximal ideal in R.

4. (10 pts) Let R be a Principal Ideal Domain and let a be a nonzero element of R. Prove that there are only finitely many maximal ideals of R containing the ideal (a) generated by a.

5. Let R be a ring and M be a left R-module. An element m in M is called a *torsion element* if there exists a nonzero element r in R such that $rm = 0$. An R-module M is called a *torsion module* if every element in M is torsion.

   (a) (3 pts) Prove that every finite abelian group is a torsion $\mathbb{Z}$-module.

   (b) (3 pts) Find an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

   (c) (4 pts) Let G be a finitely generated abelian group (not necessarily a finite group). Prove that $G \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free module over $\mathbb{Q}$.

6. For each of the following modules, determine whether or not it has the indicated properties over its designated ring. **(You must fully justify the reason for a positive conclusion and for a negative conclusion.)**

A. Free B. Projective. C. Injective. D. Flat. E. Cyclic.

(a) (4 pts) $M = \mathbb{Q}[x]/(x-1) \oplus \mathbb{Q}[x]/(x+1)$ over $\mathbb{Q}[x]$ for C, E.

(b) (4 pts) $N = \mathbb{Z}/3\mathbb{Z}$ over $\mathbb{Z}/12\mathbb{Z}$ for A, D.

(c) (2 pts) $L = \mathbb{Z}/3\mathbb{Z}$ over $\mathbb{Z}/3\mathbb{Z}$ for A, B, C.

7. Let $\zeta = \zeta_8$ be a primative $8th$ root of unity; that is, $\zeta^8 = 1$ but $\zeta^m \neq 1$ for any positive integer $m < 8$.

(a) (2 pts) Prove that $K = \mathbb{Q}(\zeta)$ is a Galois extension over $\mathbb{Q}$.

(b) (4 pts) Determine the structure of $G = \mathrm{Gal}(K/\mathbb{Q})$. Describe explicitly the automorphisms in G and their orders.

(c) (4 pts) Find the fixed field of each subgroup of G. And draw the two corresponding lattices for the field extensions $K/\mathbb{Q}$ and for the group $\mathrm{Gal}(K/\mathbb{Q})$.

8. Let θ be a root of $x^3 + x + 1$ in an extension field of $\mathbb{F}_2$.

(a) (3 pts) What is $[\mathbb{F}_2(\theta) : \mathbb{F}_2]$? (Justify your answer.)

(b) (2 pts) Prove that θ^2 is also a root of $x^3 + x + 1$.

(c) (5 pts) Is $\mathbb{F}_2(\theta)$ a Galois extension over $\mathbb{F}_2$? (Justify your answer.)