Algebra Qualifying Exam
January 7, 2021 (1pm–5pm)

For *page arrangement*, follow these general instructions closely throughout the test.

- Write in a legible fashion.
- For each sheet, write only on a single side and leave a 1×1 square inch area blank at the upper left corner – for staples.
- Start with a new sheet for each question.
- Absolutely no cell phones of any kind during the entire test.
- Arrange all papers by the order of the questions before submission.

The following instructions are given in ″Algebra Qualifying Exam January 2021 Guidelines″ under ″Exam Format″. It is the student's responsibility to read and follow these instructions carefully.

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. No calculators or other electronic devices are allowed.
4. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
5. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.
6. All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.
7. If multiple version of solutions are provided, then all versions will be graded and then the average credits is taken as the final score.
8. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
9. Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.

Exam grade: In order to pass the algebra qualifying exam, you must satisfy the following two criteria:

(a) Receive the full score from at least three questions.

(b) Receive a passing score 70% or better; equivalently, 49 points or more out of the total 70 points.

Notation. Throughout the exam, we use the following notation:

$\mathbb{Z}$ is the ring of integers;

$\mathbb{Q}$ is the field of rational numbers;

$n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$, where n is a positive integer;

$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n, where n is a positive integer.

(1) (a) (4 points) Let M be a $\mathbb{Z}$-module; i.e., M is an abelian group. Suppose that

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

is a chain of submodules such that, for $i = 1, 2, \ldots, n$, the factors M_i/M_{i-1} are simple and pairwise non-isomorphic.

Prove: If P and Q are isomorphic submodules of M, then $P = Q$.

(b) (2 points) Let $M = \mathbb{Z}$, viewed as a module over itself. Give precise examples of two submodules H_1 and H_2 of M such that $H_1 \cong H_2$ but $H_1 \neq H_2$.

(c) (4 points) Why does part (b) not violate the statement in part (a)? Justify your answer completely.

(2) (a) (3 points) Prove that no group of order $84 = 2^2 \cdot 3 \cdot 7$ is simple.

(b) (7 points) Prove that there exists an integer k_0 such that if $k \geq k_0$, then no group of order $2^k \cdot 3 \cdot 7$ is simple.

(3) Prove the following properties of a Principal Ideal Domain (PID).

(a) (5 points) Prove that any two nonzero elements of a PID have a least common multiple.

(b) (5 points) Prove that a quotient of a PID by a prime ideal is again a PID.

(4) Let R be a subring of a commutative ring S and $s \in S$. Then s is called *integral* over R if s is a root of a monic polynomial with coefficients from R, that is, there exist $a_0, \ldots, a_{n-1} \in R$ such that $s^n + a_{n-1}s^{n-1} + \cdots + a_0 = 0$.

(a) (5 points) Let A be a subring of a commutative ring B with 1. Assume that $b \in B$ is integral over A. We denote $A[b] = \{p(b) : p(x) \in A[x]\}$, where $A[x]$ is the polynomial ring over A in variable x. Prove that $A[b]$ is a finitely generated A-module.

(b) (5 points) Let B be an integral domain and A be a subring of B with 1. Assume that every element of B is integral over A. Prove that A is a field if and only if B is a field.

(5) Let $L = \mathbb{Q}(\sqrt{2}, \gamma)$, where $\gamma^2 - \gamma + 1 = 0$.

(a) (5 points) Show that L is a simple extension over $\mathbb{Q}$; i.e., show that $L = \mathbb{Q}(\alpha)$ for some $\alpha \in L$.

(b) (5 points) Let φ be a $\mathbb{Q}$-automorphism of L defined by $\varphi(\sqrt{2}) = -\sqrt{2}$ and $\varphi(\gamma) = 1 - \gamma$. Find the order of φ as an element of the Galois group of L over K. Find the fixed field of the subgroup generated by φ.

(6) Let $\mathbb{Q}$ be the field of rational numbers, and let $f(x) = x^8 + x^4 + 1$ be a polynomial in $\mathbb{Q}[x]$. Suppose F is a splitting field for $f(x)$ over $\mathbb{Q}$ and set $G = \mathrm{Gal}(F/\mathbb{Q})$.

(a) (5 points) Calculate $[F : \mathbb{Q}]$ and determine the Galois group G up to isomorphism.

(b) (5 points) Find the structure of the Galois group.

(7) By doing the following parts, you will prove that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ as $\mathbb{Z}$-modules where $d = \gcd(m, n)$ by explicitly giving the isomorphism.

(a) (3 points) Prove that the order of $(1 + m\mathbb{Z}) \otimes (1 + n\mathbb{Z})$ in $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ divides d.

(b) (3 points) Prove that there exists a $\mathbb{Z}$-module homomorphism

$$\Phi \colon \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z}$$

such that $\Phi\big((s + m\mathbb{Z}) \otimes (t + n\mathbb{Z})\big) = st + d\mathbb{Z}$.

(c) (4 points) Prove Φ from part (b) is a $\mathbb{Z}$-module isomorphism.

(8) Let U, V and W be finite-dimensional vector spaces over a field F. Suppose that

$$0 \longrightarrow U \xrightarrow{\psi} V \xrightarrow{\phi} W \longrightarrow 0$$

is a short exact sequence of F-modules.

(a) (5 points) Construct an F-module homomorphism $\mu \colon W \longrightarrow V$ such that $\phi \circ \mu$ is identity on W.

(b) (5 points) Construct an F-module homomorphism $\nu \colon V \longrightarrow U$ such that $\nu \circ \psi$ is identity on U.