Algebra Qualifying Exam
August 25, 2023 (1pm–5pm)

The following instructions are given in "Algebra Qualifying Exam August 2023 Guidelines" under "Exam Format". It is the student's responsibility to read and follow these instructions carefully.

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. No calculators or other electronic devices are allowed.
4. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
5. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.
6. All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.
7. If multiple version of solutions are provided, then all versions will be graded and then the average credits is taken as the final score.
8. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
9. Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.

Exam grade: In order to pass the algebra qualifying exam, you must satisfy the following two criteria:

(a) Receive the full score from at least three questions.

(b) Receive a passing score 70% or better; equivalently, 49 points or more out of the total 70 points.

Notation. Throughout the exam, we use the following notation:

$\mathbb{Z}$ is the ring of integers.
$\mathbb{Q}$ is the field of rational numbers.
$\mathbb{R}$ is the field of real numbers.
$\mathbb{C}$ is the field of complex numbers.
$n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$, where n is a positive integer.
$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n, where n is a positive integer.
$\mathbb{F}_p$ is a field with p elements when p is a prime, i.e. $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$.
S_n is the permutation group on n symbols.

1. (a) (2 pts) List all possible orders for the elements in S_8.
   (b) (5 pts) Find a representative for each conjugacy class of elements of all possible orders in S_8.
   (c) (3 pts) Write the statements of theorems or properties that you use in Parts (a) and (b). Describe how you use them. You do not need to prove the statements. (It is less crucial that the name of a theorem is provided, but the statement of the theorem and property must be recalled precisely.).

2. (a) (4 pts) Prove that no group of order $84 = 2^2 \cdot 3 \cdot 7$ is simple.
   (b) (6 pts) Prove that if $k \geq 20$, then no group of order $2^k \cdot 3 \cdot 7$ is simple.

3. (a) (3 pts) Find the generator of the ideal in $\mathbb{Z}$ generated by 1287 and 1452. Justify your answer.
   (b) (7 pts) Find the generator $h(x)$ of the ideal in $\mathbb{R}[x]$ generated by $f(x) = x^5 + 1$ and $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. Also find $a(x)$ and $b(x)$ in $\mathbb{R}[x]$ such that $h(x) = a(x)f(x) + b(x)g(x)$.

4. Let $R = \mathbb{C}[x]/(x^3)$.
   (a) (1 pt) Give an example of a (nonzero) zero-divisor of R. Justify your answer.
   (b) (3 pts) Let $\bar{x} = x + (x^3)$ denote the coset represented by x in R. Show that $(\bar{x})$ is a prime ideal of R.
   (c) (3 pts) Show that R has a unique maximal ideal.
   (d) (3 pts) Show that the set of zero-divisors form the unique maximal ideal of R.

5. Let V be a vector space of dimension 6 over a field K. For a linear transformation $T : V \to V$, define the *kernel sequence*

$$a(T) = (a_1, a_2, a_3, \ldots), \quad a_i(T) = \dim_K \operatorname{Ker} T^i.$$

For each of the following sequences of integers, either construct a linear transformation having this kernel sequence, or prove that such a transformation does not exist. Justify your answers.

(a) (2 pts) $(2, 4, 5, 6, 6, 6, \ldots)$
(b) (2 pts) $(0, 1, 2, 2, 2, \ldots)$
(c) (3 pts) $(1, 3, 4, 5, 5, \ldots)$
(d) (3 pts) $(2, 3, 4, 5, 5, \ldots)$

6. Let R be an integral domain with 1, and let M be an injective R-module. We recall the following helpful criterion (which is true for arbitrary commutative ring with 1):

   (*Baer's Criterion*) An R-module Q is injective if and only if for every ideal I in R any R-module homomorphism $g : I \to Q$ can be extended to an R-module homomorphism $G : R \to Q$; that is, g factors through the injection $i : I \to R$ so that $g = G \circ i$.

   (a) (5 pts) Show that M is *divisible*, in other words show that for every $m \in M$ and every nonzero $r \in R$ there exists $n \in M$ such that $nr = m$.
   (b) (5 pts) Let P be a torsion R-module, in other words assume that for every $p \in P$ there exists a nonzero $r \in R$ such that $pr = 0$. Show that $M \otimes_R P = 0$.

7. Identify the Galois groups of each of the following extensions of $\mathbb{Q}$. Justify your answers.

   (a) (5 pts) $\mathbb{Q}(\sqrt{2} + \sqrt{-1})$.
   (b) (5 pts) The splitting field of the polynomial $x^4 - 8x^2 + 9$.

8. Let $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$ be a finite extension such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ and $\mathbb{Q}(\beta)/\mathbb{Q}$ are Galois and $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q}$. Let G denote the Galois group of $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}(\alpha + \beta)$.

   (a) Prove that $\sigma(\alpha) - \alpha = \beta - \sigma(\beta)$ for all $\sigma \in G$.
   (b) For each $\sigma \in G$, prove that there exists $r \in \mathbb{Q}$ (apriori depends on σ) such that $\sigma(\alpha) = \alpha + r$ and $\sigma(\beta) = \beta - r$.
   (c) Let $\sigma \in G$. Prove the following:
      i. $\sigma^k(\alpha) = \alpha + kr$ for $k \in \mathbb{N}$ and $r \in \mathbb{Q}$ as above in (b).
      ii. $\sigma(\alpha) = \alpha$ and $\sigma(\beta) = \beta$.

      (Hint: σ is of finite order in G.)
   (d) Prove that $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha + \beta)$.