Algebra Qualifying Exam
January 6, 2022 (1pm–5pm)

For *page arrangement*, follow these general instructions closely throughout the test.

- Write in a legible fashion.
- For each sheet, write only on a single side and leave a 1×1 square inch area blank at the upper left corner – for staples.
- Start with a new sheet for each question.
- Absolutely no cell phones of any kind during the entire test.
- Arrange all papers by the order of the questions before submission.

The following instructions are given in ″Algebra Qualifying Exam January 2022 Guidelines″ under ″Exam Format″. It is the student's responsibility to read and follow these instructions carefully.

(1) Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.

(2) If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.

(3) No calculators or other electronic devices are allowed.

(4) Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.

(5) Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.

(6) All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.

(7) If multiple version of solutions are provided, then all versions will be graded and then the average credits is taken as the final score.

(8) No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

(9) Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.

Exam grade: In order to pass the algebra qualifying exam, you must satisfy the following two criteria:

(a) Receive the full score from at least three questions.

(b) Receive a passing score 70% or better; equivalently, 49 points or more out of the total 70 points.

Notation. Throughout the exam, we use the following notation:

$\mathbb{Z}$ is the ring of integers;
$\mathbb{Q}$ is the field of rational numbers;
$\mathbb{R}$ is the field of real numbers;
$\mathbb{C}$ is the field of complex numbers.
$n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$, where n is a positive integer;
$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n, where n is a positive integer.

1. (a) (3 pts) Give an example of a Sylow 2-subgroup of A_4. Is it a unique Sylow 2-subgroup of A_4? Justify your answers.
   (b) (2 pts) Give an example of a Sylow 3-subgroup of A_4. Is it a unique Sylow 3-subgroup of A_4? Justify your answers.
   (c) (2 pts) Is the cyclic subgroup of A_5 generated by the 5-cycle (12345) a Sylow 5-subgroup? Justify your answer.
   (d) (3 pts) How many distinct Sylow 5-subgroups are there in A_5? Justify your answer.

2. (a) (5 pts) Let the group

$$GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

   act naturally by matrix multiplication on $\mathbb{R}^2$, viewed as the set of column vectors.
   (i) Prove that the orbit of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^2$ is $\mathbb{R}^2 \setminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.
   (ii) Find the stabilizer of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (b) (5 pts) Let the group

$$GL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}$$

   act naturally by matrix multiplication on $\mathbb{Z}^2$, viewed as the set of column vectors.
   (i) Prove that the orbit of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{Z}^2$ is

$$\left\{ \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{Z}^2 : \gcd(m, n) = 1 \right\}.$$

   (ii) Find the stabilizer of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ under this action.

3. Let R be a Euclidean domain with norm N. The goal of Part (b) of this question is to prove that R is a PID.

   (a) (5 pts) Let
   $$m = \min\{N(x) : x \in R \backslash \{0\}\},$$
   and let $y \in R$ be an element such that $N(y) = m$. Prove that y is a unit.

   (b) (5 pts) Let $I \subset R$ be a nonzero ideal, and let
   $$m_I = \min\{N(x) : x \in I \backslash \{0\}\}.$$
   Prove that $I = (a)$, where $a \in I$ is such that $N(a) = m_I$.

4. Define a module to be *irreducible* if it does not have a nonzero proper submodule. In each of the following cases, determine whether the module is irreducible over the corresponding ring. Justify your answers.

   (a) (2.5 pts) $M = \mathbb{Z}[x]/(x^4 + 1)$ over $R = \mathbb{Z}[x]$.

   (b) (2.5 pts) $N = \mathbb{Q}[x]/(x^4 + 1)$ over $S = \mathbb{Q}[x]$.

   (c) (2.5 pts) $P = \mathbb{R}[x]/(x^4 - 1)$ over $T = \mathbb{R}[x]$.

   (d) (2.5 pts) $Q = \mathbb{C}[x]/(x^4 + 1)$ over $A = \mathbb{C}[x]$.

5. (10 pts) Consider the following three matrices, which all have the same characteristic polynomial: $(x - 2)^2(x - 3)$.
   $$A = \begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$
   Determine the similarity among the three matrices by determining their list of invariant factors or elementary divisors. Find the Jordan canonical forms for all three matrices.

6. Let R be a commutative ring with unity 1, and let I and J be ideals of R. The quotients R/I and R/J have R-module structures defined by
   $$r(a + I) = ra + I, \quad r(a + J) = ra + J, \quad r, a \in R.$$

   (a) (4 pts) Prove that there exists an R-module homomorphism
   $$\phi : R/I \otimes_R R/J \to R/(I + J)$$
   such that
   $$\phi\left((a + I) \otimes (b + J)\right) = ab + (I + J)$$
   for all $a, b \in R$.

   (b) (3 pts) Prove that every element of $R/I \otimes_R R/J$ has the form $(1 + I) \otimes (t + J)$ for some $t \in R$ (you don't need part (a) to prove this).

   (c) (3 pts) Prove that the homomorphism ϕ from part (a) is an isomorphism.

7. (a) (2 pts) Find the minimal polynomial of $\sqrt{2+\sqrt{2}}$ over $\mathbb{Q}$.

   (b) (5 pts) Prove that $E = \mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a Galois extension of $\mathbb{Q}$, and determine the Galois group Gal(E/$\mathbb{Q}$).

   (c) (3 pts) Draw the lattice of subgroups of the Galois group Gal(E/$\mathbb{Q}$) and that of their corresponding fixed fields.

8. Let $\mathbb{F}_{11}$ denote the ring $\mathbb{Z}/11\mathbb{Z}$.

   (a) (1 pt) Prove that $\mathbb{F}_{11}$ is a field.

   (b) (1 pt) Prove that the polynomials x^2+1 and x^2+2x+2 are both irreducible over $\mathbb{F}_{11}$.

   (c) (4 pts) Prove that the two quotient rings

$$K_1 = \mathbb{F}_{11}[x]/(x^2+1) \quad \text{and} \quad K_2 = \mathbb{F}_{11}[y]/(y^2+2y+2)$$

   are both fields with 121 elements.

   (d) (4 pts) Let p be a polynomial with coefficients in $\mathbb{F}_{11}$. Prove that the map

$$f : K_1 \longrightarrow K_2$$

   that sends the element $p(\bar{x})$ of K_1 to $p(\bar{y}+\bar{1})$ of K_2 is well-defined and gives a ring (hence a field) homomorphism from K_1 to K_2. Deduce that K_1 and K_2 are isomorphic fields.