Algebra Qualifying Exam
January 6, 2023 (1pm–5pm)

For *page arrangement*, follow these general instructions closely throughout the test.

- Write in a legible fashion.
- For each sheet, write only on a single side and leave a 1×1 square inch area blank at the upper left corner – for staples.
- Start with a new sheet for each question.
- Absolutely no cell phones of any kind during the entire test.
- Arrange all papers by the order of the questions before submission.

The following instructions are given in ″Algebra Qualifying Exam January 2021 Guidelines″ under ″Exam Format″. It is the student's responsibility to read and follow these instructions carefully.

1. Do seven problems out of eight. The exam is 70 points. Each problem is worth 10 points.
2. If you attempt solutions to all eight problems, then only the **first seven** submitted problems will be graded.
3. No calculators or other electronic devices are allowed.
4. Write in a legible fashion. Give proper mathematical justification of all your statements. Your solutions must be detailed enough to get full credit.
5. Start with a new sheet for each problem. Clearly write the problem number and your name before beginning your solution. For each sheet, write only on a single side. Arrange all papers by the order of the questions before submission.
6. All your submitted written responses will be graded. If you write any statements (correct or incorrect) which are not used in your solution, you will be penalized. Please make sure to erase clearly the responses you don't want to be graded.
7. If multiple version of solutions are provided, then all versions will be graded and then the average credits is taken as the final score.
8. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.
9. Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.

Exam grade: In order to pass the algebra qualifying exam, you must satisfy the following two criteria:

(a) Receive the full score from at least three questions.

(b) Receive a passing score 70% or better; equivalently, 49 points or more out of the total 70 points.

Notation. Throughout the exam, we use the following notation:

$\mathbb{Z}$ is the ring of integers.
$\mathbb{Q}$ is the field of rational numbers.
$\mathbb{R}$ is the field of real numbers.
$\mathbb{C}$ is the field of complex numbers.
$n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$, where n is a positive integer.
$\mathbb{Z}/n\mathbb{Z}$ is the ring of residue classes of integers modulo n, where n is a positive integer.
$\mathbb{F}_q$ is the field with q elements, where $q = p^k$ for some prime p and $k \geq 1$; $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$.

1. Let G be a group of order 72. Let A be the set of all subgroups of order 9. We know that by Sylow's theorem, A is not an empty set.

   (a) (5 points) Prove that there exists a group homomorphism
   $$\varphi : G \longrightarrow S_A$$
   from G to the permutation group on A.

   (b) (5 points) Prove that a group of order 72 is not simple.

2. Let G denote a group. For each $g \in G$, we define $\phi_g : G \to G$ by $\phi_g(x) = gxg^{-1}$. We call ϕ_g an *inner automorphism* of G. The group of all automorphisms of G is denoted by $\mathrm{Aut}(G)$ and its subgroup of all inner automorphisms of G is denoted by $\mathrm{Inn}(G)$.

   (a) (2 points) Prove that $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$ where $Z(G)$ is the center of G.

   (b) (2 points) Prove that if $\mathrm{Aut}(G)$ is trivial, then G is abelian.

   (c) (2 points) Prove that if $\mathrm{Aut}(G)$ is trivial, then the order of all nonidentity elements in G is 2.

   (d) (2 points) If G is abelian and the order of all nonidentity elements in G is 2, then we can view G as a $\mathbb{Z}/2\mathbb{Z}$-module or equivalently a vector space over the field $\mathbb{Z}/2\mathbb{Z}$. If the dimension of G over $\mathbb{Z}/2\mathbb{Z}$ is larger than 1, prove that $\mathrm{Aut}(G)$ is non trivial.

   (e) (2 points) Prove that if $\mathrm{Aut}(G)$ is trivial, then G is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

3. (a) (5 points) Let R be a commutative ring. Let I and J be ideals of R. Assume P is a prime ideal of R that contains IJ. Prove either I or J is contained in P.

   (b) (5 points) Give a counterexample of explicit I, J and P in the polynomial ring over a field of one or two variables, i.e. $R = k[x]$ or $k[x, y]$, that shows that the above assertion does not hold if P is not a prime ideal.

4. Let R be commutative and let I be an ideal in R. Define

$$\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n > 0 \text{ in } \mathbb{Z}\ \}$$

(a) (3 points) Prove that $\sqrt{I}$ is an ideal in R containing I.

(b) (2 points) Prove that if I is a prime ideal, then $\sqrt{I} = I$.

(c) (2 poitns) Let $R = \mathbb{C}[x, y]$. Assume that I is a proper nonzero ideal in R. Let $(a, b) \in \mathbb{C} \times \mathbb{C}$. Prove that $g(a, b) = 0$ for all $g \in I$ if and only if $f(a, b) = 0$ for all $f \in \sqrt{I}$.

(d) (3 points) Consider the ideal $I = (x^4 - 2x^2y^2 + y^4, x^4 - y^4)$ in $k[x, y]$ where $k = \mathbb{R}$. Prove that $\sqrt{I}$ is a principal ideal and find a generator for it. Solve the same problem for $k = \mathbb{F}_2$.

5. Let $\alpha = \sqrt{1 + \sqrt[3]{2}} \in \mathbb{C}$.

(a) (1 point) Show that the degree of $\mathbb{Q}(\alpha^2)$ over $\mathbb{Q}$ is 3.

(b) (2 points) Find the minimal polynomial of α over $\mathbb{Q}$. Justify your answer.

(c) (2 poins) Let K denote the splitting field of the minimal polynomial of α over $\mathbb{Q}$ and let G denote the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. Prove that K contains the splitting field of $x^3 - 2$ over $\mathbb{Q}$.

(d) (2 points) Prove that the Galois group of the splitting field of $x^3 - 2$ over $\mathbb{Q}$ is S_3.

(e) (3 points) Let G be as in Part (c) above. Prove that G has a normal subgroup H such that the quotient G/H is isomorphic to S_3.

6. For each of the following extensions $\mathbb{Q} \subset K$, prove that it is Galois, find the Galois group, and find all intermediate quadratic extensions $\mathbb{Q} \subset F \subset K$:

(a) (3 points) $K = \mathbb{Q}\left(\sqrt{2 + \sqrt{2}}\right)$.

(b) (4 points) $K = \mathbb{Q}\left(\sqrt{7 + 2\sqrt{10}}\right)$.

(c) (3 points) $K = \mathbb{Q}\left(\sqrt{4 + 2\sqrt{3}}\right)$.

7. Let G be an abelian group so it has a natural $\mathbb{Z}$-module structure.

(a) (5 points) Prove that all nonzero elements in $G \otimes_{\mathbb{Z}} \mathbb{Q}$ have inifinite order.

(b) (5 points) Give precise examples for G such that $G \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $G \otimes_{\mathbb{Z}} \mathbb{Q}$ is not trivial, respectively.

8. For each of the following R-modules M, verify wether M has each of the following properties:

free, projective, flat, injective.

(a) (3 points) $M = \mathbb{Z}/p\mathbb{Z}$ over $R = \mathbb{Z}$.

(b) (4 points) M is the ideal $(2, x)$ in the ring $R = \mathbb{Z}[x]$.

(c) (3 points) $M = \mathbb{C}$ over $R = \mathbb{R}$.