Central Michigan University
Department of Mathematics
Analysis Qualifying Exam
August 25, 2017

Instructions

1. The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).
2. You will be required to complete five problems from the Real Analysis (MTH 632) part, and another five problems from the Complex Analysis (MTH 636) part.
3. In either section, if you attempt solutions to all six problems, then only the first five problems will be graded.
4. Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.
5. No calculators or other electronic devices are allowed.
6. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.
7. Give proper mathematical justification of all your statements.
8. Each question is worth 10 points. For questions divided into multiple subquestions, the points for each part are indicated.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(C1) Suppose that $f = u + iv$ is analytic in a domain D, with u and v real-valued functions. Which of the following functions are analytic in D for every analytic function f? Why?

(a) [5 points] $f_1 = (u^2 - v^2) + 2iuv$

(b) [5 points] $f_2 = u - iv$

(C2) (a) [5 points] Let f be entire and suppose that the sixth derivative $f^{(6)}$ of f is bounded on the whole complex plane. Prove that f is a polynomial of degree at most 6.

(b) [5 points] Let f be entire and suppose there is an irrational number $a \in \mathbb{R} \setminus \mathbb{Q}$ such that for all $z \in \mathbb{C}$

$$f(z+1) = f(z) = f(z+a).$$

Prove that f is constant. Hint: consider the set $\{m + na | n, m \in \mathbb{Z}, 0 < m + na < 1\}$.

(C3) (a) [5 points] Find a Laurent series for

$$f(z) = \frac{1}{(z-1)^2(z+1)^2}$$

valid for $|z| > 1$.

(b) [5 points] For the function

$$f(z) = \frac{z^3+1}{z^2(z+1)}$$

find and classify all isolated singularities in $\mathbb{C}$. As well, find the order of every pole.

(C4) (a) [5 points] Let f be analytic in a domain D with $z_0 \in D$ and $f'(z_0) \neq 0$. Prove that

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{1}{f(z) - f(z_0)} dz,$$

where C is a circle of sufficiently small radius centered at z_0 and oriented counterclockwise.

(b) [5 points] Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2 - x + 1}\, dx$

(C5) Let f be a function analytic inside and on the unit circle. Suppose that

$$|f(z) - z| < 1$$

on the unit circle.

(a) [5 points] Show that f has precisely one zero inside of the unit circle.

(b) [5 points] Show that $|f'(\frac{1}{2})| \leq 4$.

(C6) Let f be a nonconstant function that is analytic and nonzero on a bounded domain D, and continuous on $\overline{D}$. Show that the minimum value of $|f(z)|$ must occur on ∂D.

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) Consider function f on the interval $[0,1]$ defined by

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in [0, \frac{1}{2}] \setminus \mathbb{Q} \\ \cos(\pi x) & \text{if } x \in [\frac{1}{2}, 1] \setminus \mathbb{Q} \\ e^{-x^2} + 2 & \text{if } x \in \mathbb{Q} \cap [0,1], \end{cases}$$

where as usual $\mathbb{Q}$ is the set of rational numbers.

(a) [3 points] What is the set of points where f is continuous?

(b) [3 points] Show that f is measurable.

(c) [4 points] Compute the Lebesgue integral

$$\int_{[0,1]} f.$$

(R2) For each one of the following, either give an example of the object described, or state that such an object does not exist. If such an object does not exist, justify your answer in at most 3 short sentences.

(a) [4 points] A sequence $\{f_n\}$ of nonnegative measurable functions on $[0,1]$, such that

$$\lim_{n\to\infty} f_n(x) = x^2 \text{ for each } x \in [0,1],$$

and

$$\lim_{n\to\infty} \int_{[0,1]} f_n = \frac{1}{\pi}.$$

(b) [3 points] A continuous non-negative function f in $L^1(\mathbb{R})$ such that the series

$$\sum_{n=1}^{\infty} f(n)$$

is divergent.

(c) [3 points] A measurable function f on $\mathbb{R}$ such that $f \in L^2(\mathbb{R})$, $f \notin L^1(\mathbb{R})$ and $f \notin L^3(\mathbb{R})$

(R3) Let $g \in L^1(\mathbb{R})$, and define a function f on $\mathbb{R}$ by setting:

$$f(x) = \int_{\mathbb{R}} g(y)\cos(xy)dy$$

where the integral on the right is a Lebesgue integral.

(a) [2 points] Why does the integral on the right hand side exist as a finite real number for each x?

(b) [3 points] Show that

$$\sup_{x\in\mathbb{R}} |f(x)| \leq \|g\|_1 .$$

(c) [5 points] Show that f is a continuous function on $\mathbb{R}$.

(R4) Let $f \in L^\infty([0,\pi])$. Show that the limit

$$\lim_{p\to\infty} \|f\|_p$$

exists, and find the value of this limit. (Here p tends to infinity through all real numbers).

(R5) (a) [5 points] For each positive integer k, let ψ_k be a real-valued continuous function on the interval $[0,1]$ such that the sequence $\{\psi_k\}$ is orthogonal in the Hilbert space $L^2([0,1])$. Suppose that $\psi_1 > 0$ on $(0,1)$. Show that for each $k \geq 2$, there is an $x_k \in (0,1)$ such that $\psi_k(x_k) = 0$.

(b) [5 points] Let $f \in L^2(\mathbb{R})$ and suppose that $\{\phi_k\}$ is an orthonormal sequence in $L^2(\mathbb{R})$. Show that the series

$$\sum_{k=1}^{\infty} \frac{2k}{k+3} \cdot \langle f, \phi_k \rangle \cdot \phi_k$$

converges in $L^2(\mathbb{R})$, where $\langle g, h\rangle = \int_{\mathbb{R}} g \cdot \overline{h}$ is the standard inner product of $L^2(\mathbb{R})$.

(R6) For each one of the following, either give an example of the object described, or state that such an object does not exist. If such an object does not exist, justify your answer in at most 3 short sentences.

(a) [4 points] A monotone function $f : [0,1] \to \mathbb{R}$ that is discontinuous on $\mathbb{Q} \cap (0,1)$.

(b) [3 points] Two measurable functions f and g on $[0,1]$ such that

$$\int_{[0,1]} |f|^3 = 64, \int_{[0,1]} |g|^{\frac{3}{2}} = \frac{1}{8}, \int_{[0,1]} fg = -2.$$

(c) [3 points] A non-measurable function $f : \mathbb{R} \to (0,\infty)$ such that the function $\log f$ is measurable.