Analysis Qualifying Exam August 24, 1996, 8:00-12:00

Please put solutions to each problem on separate sheets.

<u>Section A</u> Answer <u>10</u> of the 12.

1. Let $A \subset \mathbb{R}^n$. Suppose every sequence in A has a subsequence converging to a point in A. Use the open covering definition of compactness to prove that A is compact.

2. Suppose $\{f_n\}$ and $\{g_n\}$ are sequence of real-valued functions defined on a set $E \subset \mathbb{R}$. Assume $\{f_n\}$ and $\{g_n\}$ converge uniformly to functions f and g, respectively. Discuss the convergence properties of the sequences $\{f_n \pm g_n\}$, $\{f_n g_n\}$, and $\{f_n/g_n\}$. Include proofs and counterexamples where necessary.

3. Let $F(x,y) = (e^x \cos y, e^x \sin y)$.

   (a) Explain why the Inverse Function Theorem is applicable to F for all $(x,y) \in \mathbb{R}^2$.

   (b) Is F one-to-one? Prove your answer.

   (c) Does part (b) contradict part (a)? Prove your answer.

4. Let U be an open set in $\mathbb{R}^2$ whose boundary ∂A is a closed simple smooth curve. Suppose $u, v : V \to \mathbb{R}$ are twice continuously differentiable on V, V an open set containing $U \cup \partial U$. Show that
$$\int_{\partial U} \left[u\frac{\partial v}{\partial x}\,dx + u\frac{\partial v}{\partial y}\,dy \right] = \iint_U J(u,v)\,dxdy$$
where $J(u,v) = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$

5. Assume f and f' are in $L^1(\mathbb{R})$ and that f is absolutely continuous on each $[a,b] \subset \mathbb{R}$. Show that $f(x) \to 0$ as $x \to \infty$.

6. A set $E \subset \mathbb{R}$ has content zero if for all $\epsilon > 0$ there is a finite collection of intervals $\{I_j\}_{j=1}^n$ such that $E \subset \cup_{j=1}^n I_j$ and $m(\cup_{j=1}^n I_j) < \epsilon$.

   (a) Prove that if E has content zero then E has measure zero.

   (b) Suppose E has measure zero, does it necessarily follow that E has content zero? Prove or find a counterexample.

   (c) Give a class of sets where content zero and measure zero coincide. Does this class of sets comprise a measure space? Prove your answer.

7. (a) Let $f : \mathbb{R} \to \mathbb{R}$. Show that the set of points of continuity of f is a $\mathcal{G}_\delta$ set.

   (b) Is the set of rational numbers a $\mathcal{G}_\delta$ set? Prove your answer.

   (c) Is there a function continuous at every rational point and discontinuous at every irrational point? Prove your answer.

8. In each assume Lebesgue measure on $\mathbb{R}$.

   (a) Prove that $L^2([a,b]) \subset L^1([a,b])$.

(b) Prove that $L^2([0,\infty)) \not\subset L^1([0,\infty))$ by giving an explicit example.

9. Suppose $\{f_n\}$ is a sequence of entire functions that converge uniformly to zero on $\mathbb{C}$. Prove that all but a finite number of the f_n are constants. Does this remain true if uniform convergence is replaced with pointwise convergence? Prove your answer.

10. Show that $\int_{|z|=1} |z|\, dz = 0$. Does Morera's Theorem imply that $|z|$ is analytic at $z = 0$? Explain.

11. Consider the function $f(z) = \dfrac{2z^2 - 1}{z^4 + 5z^2 + 4}$.

   (a) Find the singularities of f and classify them.

   (b) Calculate the residues of f at its singular points.

   (c) Show that for large R, $|f(z)| \le \dfrac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)}$.

   (d) Show that the integral

$$\int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx = \frac{\pi}{4}$$

   by integrating around an appropriate closed curve depending on R and letting $R \to \infty$.

12. Let $a > e$. Show that the equation $e^z = az^n$ has n roots inside the unit circle. When $n = 2$, show that both roots are real.

Section B Select one of the two parts

Part 1 Answer 2 of the three

1. (a) What is meant by $L^p(\mu)$ where $1 \le p < \infty$ and μ is a σ-finite measure, so that $(X, \mathcal{B}, \mu)$ is a measure space.

   (b) Suppose F is a bounded linear functional on $L^p(\mu)$, where $\mu(X) < \infty$. Define a function ν on $\mathcal{B}$ by $\nu(E) = F(\chi_E)$. Show that ν is a signed measure absolutely continuous with respect to μ. Argue that there must therefore be an integrable function g so that $\nu(E) = \int_E g\, d\mu$.

   (c) Show that $F(\phi) = \int \phi g\, d\mu$ for all simple functions and then argue that $F(f) = \int fg\, d\mu$ for all $f \in L^p(\mu)$.

   (d) What can you say about g? (You needn't prove your statement.)

2. (a) What is meant by saying that a measure space $(X, \mathcal{B}, \mu)$ is complete.

   (b) Given a measure space $(X, \mathcal{B}, \mu)$, let

$$\mathcal{B}_0 = \{E : E = A \cup B, \text{ where } B \in \mathcal{B} \text{ and } A \text{ is a subset of some } C \in \mathcal{B} \text{ with } \mu(C) = 0\}.$$

   Show $\mathcal{B}_0$ is a σ-algebra which contains $\mathcal{B}$.

(c) Define a measure μ_0 on $\mathcal{B}_0$ so that $\mu_0(E) = \mu(D)$ if $E \in \mathcal{B}$. Show that μ_0 is a measure and $(X, \mathcal{B}_0, \mu_0)$ is a complete measure space.

3. The convolution of $\phi * f$ of two functions ϕ and f defined on $\mathbb{R}$ is the function

$$(\phi * f)(x) = \int_{\mathbb{R}} \phi(y) f(x-y)\, dy.$$

If ϕ is a nonnegative continuous function with $\int_{\mathbb{R}} \phi(y)\, dy = 1$ and $f \in L^\infty(\mathbb{R})$, then show that

$$[(\phi * f)(x)]^2 \le (\phi * [f^2])(x)$$

for every $x \in \mathbb{R}$.

<u>Part 2</u> Answer <u>2</u> of 3

1. (a) State the Riemann Mapping Theorem.
   (b) Prove that the complex plane is only conformally equivalent to itself.
   (c) Show that the only automorphism f of the unit disk with $f(0) = 0$, $f'(0) > 0$ is the identity map $f(z) \equiv z$.

2. (a) Prove that a nonconstant harmonic function cannot attain a maximum or minimum in a domain.
   (b) If $u(z) = u(x, y)$ is harmonic in the plane with $u(z) \le |z|^n$ for every z, show that $u(z)$ is a polynomial in the two variables x and y.

3. (a) Let $f(z) = \int_0^\infty t^3 e^{-zt}\, dt$ be analytic at all points z for which $\operatorname{Re} z > 0$. Find a function $F(z)$ which is the analytic continuation of $f(z)$ into the half plane $\operatorname{Re} z < 0$.
   (b) Prove that a function is meromorphic if and only if it can be expressed as a quotient of entire functions.