Analysis Qualifying Exam **August 28, 1998** **9:00-1:00** **PE 223**

Section A: Do **ten** of the **twelve.**

1. Consider the series $\sum_{n=1}^{\infty} a_n \sin nx$ where $|a_n| \le 1/n^2$ for every n.

a.) For what values of x does the series converge?

b.) Is the convergence of the series uniform on the set where it converges? Justify your answer.

c.) Given that $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n \end{cases}$ evaluate $\int_{-\pi}^{\pi} \sin mx \left(\sum_{n=1}^{\infty} a_n \sin nx \right) dx$ where m is a fixed positive integer. (Does term-by-term integration work?)

d.) Suppose $f(x) = \sum_{n=1}^{\infty} a_n \sin nx \, dx$ for $x \in [-\pi, \pi]$. Write a_n in terms of an integral involving f(x). What kind of series is this?

2. a.) Indicate whether the following statement is true or false:
If $\{f_n(x)\}$ is a sequence of continuous functions on [a, b] and $f_n(x) \to f(x)$ pointwise on [a, b], then f is continuous. If false, give an example to justify your answer, and indicate how to alter the statement so that it is true.

b.) Repeat the analysis in part a.) as it pertains to the following statement: If $\{f_n(x)\}$ is a sequence of continuous functions converging pointwise to f(x) on [a, b], then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$.

c.) For each n, let $f_n(x) = \dfrac{x}{1+nx^2}$ for $-1 \le x \le 1$. Show that $f_n(x) \to 0$ uniformly on [-1, 1]. (Hint: determine the max and min values of f_n on [-1, 1]). Is it true that $\lim_{n\to\infty} f_n'(x) = 0$? How do you explain this?

3. a.) Show that the unit ball $U = \left\{ x \in \Re^n : \sum_{j=1}^{n} |a_j|^2 \le 1 \right\}$ in $\Re^n$, where $x = (a_1, a_2, \ldots, a_n)$, is sequentially compact. Prove this directly - You may use, without proof, the Bolzano-Weierstrass Theorem in $\Re$; that is, every bounded sequence of real numbers has a convergent subsequence.

b.) Prove that any continuous real valued function defined on U must be bounded and, in fact, take on its maximum value. Prove this using the result of part a.).

4. Let $F=(f_1,f_2):\Re^5 \to \Re^2$ be given by

$$f_1(x_1,x_2,x_3,x_4,x_5)=x_1-3x_2+2x_3-x_4-x_5$$
$$f_1(x_1,x_2,x_3,x_4,x_5)=2x_1+x_2-5x_3+x_4+2x_5.$$

a.) What does it mean to say that a function such as F above is differentiable at a given point?

b.) Is the function F above differentiable at $\overline{0}=(0,0,0,0,0)$? How do you know? (You may quote pertinent theorems if you wish.) Is F differentiable at every point of $\Re^5$?

c.) If the answer to part b.) is yes, compute the derivative of F at 0.

d.) Can the system $F(x_1,x_2,x_3,x_4,x_5)=\overline{0}$ be solved for x_1,x_2 in terms of x_3,x_4,x_5? Are there differentiable functions g_1,g_2 on an open set in $\Re^3$ so that $x_1=g_1(x_3,x_4,x_5)$ and $x_2=g_2(x_3,x_4,x_5)$? What Theorem guarantees answers to these questions and how does it apply?

5. a.) Describe the development of Lebesgue measure and the Lebesgue integral, starting with the definition of outer measure and concluding with the important convergence theorems. State carefully the pertinent definitions and theorems.

b.) Give an example to show the use of one of the important convergence theorems to establish the Lebesgue integrability of a function.

6. a.) Let f be an extended real valued function whose domain D is Lebesgue measurable. Show that $\{x\in D: f(x)<\alpha\}$ is measurable for all $\alpha\in\Re$ if and only if $\{x\in D: f(x)\le\alpha\}$ is measurable for all α.

b.) Suppose $\{f_n\}$ is a sequence of measurable functions each defined on the measurable set D. Prove that $g(x)=\sup_n f_n(x)$ is also measurable.

c.) Let f be a non-negative measurable function. Show that there is an increasing sequence $\{p_n\}$ of nonnegative simple functions, each of which vanishes outside a set of finite measure, such that $f=\lim p_n$.

7. a.) Suppose f is a continuous function defined on a closed interval [a, b]. Does f have a finite Lebesgue integral on [a, b]? How does this compare to the Riemann integral of f on [a, b].

b.) Calculate $\int_0^1 \frac{dx}{(1-x)^{1/3}}$ as a Lebesgue integral. Identify any Theorems used in the calculation of this integral.

8. a.) Define what it means for a real valued function f defined an [a, b] to be absolutely continuous. Give an example of such a function.

b.) Assume that f is Lebesgue integral an [a, b]. Define f on [a, b] by $F(x) = \int_a^x f(t)\,dt$.

Prove that F is of bounded variation on [a, b].

c.) What other properties can you attribute to the function F of part b.)?

d.) Suppose $g(x) = \sum |f_k(x)| < +\infty$ for all $x \in [0,1]$, $\int g^p < +\infty$, and $s(x) = \sum f_k(x)$, where each f_k is measurable and $\int_0^1 |f_k|^p < \infty$. Let $s_n(x) = \sum_{k=1}^{n} f_k(x)$. Prove that $\int |s|^p < \infty$, and $\int_0^1 |s_n - s|^p \to 0$ as $n \to \infty$. (Note that $|s_n(x) - s(x)|^p \le 2^p [g(x)]^p$)

9. a.) Show that $\int_C \frac{\sin z}{4z+\pi}\,dz = -\frac{\sqrt{2}}{4}\pi i$ where C is the circle $|z| = 1$ with positive orientation.

b.) Show that $\int_C \frac{e^{z^2}}{(z-i)^4}\,dz = \frac{-4\pi}{3e}$ where C is the circle $|z| = 2$ with positive orientation.

10. a.) State and prove Liouville's Theorem.

b.) Let f be entire and suppose that $f^{(5)}(z)$ is bounded in the complex plane. Prove that f must be a polynomial of degree at most 5.

11. Find the Laurent series for $\frac{1}{(z-1)(z+2)}$

a.) for $1 < |z| < 2$

b.) for $|z| > 2$

12. a.) State Rouche's Theorem.

b.) How many roots does the equation $z^4 + 8z^3 - 3z - 8 = 0$ have in $|z| < 2$? Justify your answer.

Section B: Select one of the two parts.

Part 1. Answer two of the three

1. a.) What is meant by saying that (Ω, Σ, μ) is a measure space?

b.) For a given set E, let $c(E)$ denote the number of elements in E. For a given set S, is $(S, 2^S, c)$ an example of a measure space? If so, is it a finite measure space? σ-finite?

c.) Consider the set N of positive integers, then $(N, 2^N, c)$ is a measure space. Let $f(n) = \frac{1}{2^n}$ for each $n \in N$. Is f a measurable function? Calculate $\int_N f(n)\,dc(n)$, using appropriate theorems.

2. a.) State the Radon-Nikodym Theorem.

b.) What do we mean by $L^P(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a σ-finite measure space?

c.) Let F be a bounded linear functional on $L^P(\Omega, \Sigma, \mu)$ where μ is a finite measure. Define a set function γ on Σ by

$$\gamma(E) = F(\chi_E).$$

Show that γ is a signed measure which is absolutely continuous with respect to μ. Does it follow that there exists a measurable function g such that $\gamma(E) = \int_E g\,d\mu$ for every $E \in \Sigma$?

3. a.) Suppose (X, A, μ), (Y, β, γ) are two complete measure spaces. What do we mean by the product measure $\mu x \gamma$ on $X \times Y$? Give some indication of how this is developed.

b.) State Fubini's Theorem.

c.) Let $X = Y = N$ (the positive integers) and $\mu = \gamma$ be the counting measure.

Let $f(x,y)=\begin{cases} 2-2^{-x} & \text{if } x=y \\ -2+2^{-x} & \text{if } x=y+1. \\ 0 & \text{otherwise} \end{cases}$

Does the conclusion of Fubini's Theorem hold for this function. Justify your answer.

Part 2: Answer two of the three.

1. Suppose $\{U_n(z)\}$ is a sequence of harmonic functions that converges uniformly on all compact subsets of a domain D to a function $U(z)$. Then, prove that $U(z)$ is harmonic throughout D.

2. If f(z) is analytic in the disk $|z-z_0|<R$ and the Taylor series expansion of f(z) about $z=z_0$ has a radius of convergence R then, prove that f(z) has at least one singular point on the circle $|z-z_0|=R$.

3. a.) Let $f(z)=\int_0^\infty t^3 e^{-zt}\,dt$ be analytic at all points z for which Re $z>0$. Find a function F(z) which is the analytic continuation of f(z) into the half-plane Re $z<0$.

   b.) Prove that a function is meromorphic if and only if it can be expressed as a quotient of entire functions.