Analysis Qualifying Exam **August 29, 2003, 9:00-1:00**

Section A. Do both parts.

Part 1 Do (only) *five* of the six problems.

1. Provide a discussion of the Riemann integral including some of its properties and its inadequacies. Give pertinent definitions. Show how the Lebesgue generalization is defined and how it overcomes some of the inadequacies of the Riemann integral. Include definitions and significant theorems. Discuss the key differences in the definitions.

2. (a) Is it true or is it false that $\int_E f d\lambda = 0$ implies that $f = 0$ a.e. where λ denotes Lebesgue measure on a Lebesgue measurable set E and f is a Lebesgue measurable function? If true, prove it; if false, correct the statement so that it is true and then prove it.

   (b) Prove that $\|f\| = \int_0^1 |f| d\lambda$ defines a norm on $L^1[0,1]$.

3. Suppose that f is a measurable function defined on a measure space $(X, \mathcal{M}, \mu)$. Show that there exists a sequence $\{\sigma_n\}$ of simple functions such that

$$\lim_{n\to\infty} \sigma_n(x) = f(x) \text{ for all } x \in X.$$

   Can you choose the σ_n so that $|\sigma_n(x)| \leq |f(x)|$ for each x? What more can be said if you know that f is nonnegative? Bounded? Finally, if f is integrable, is it true that

$$\int_X f d\mu = \lim_{n\to\infty} \int_X \sigma_n d\mu?$$

   Justify your answer.

4. (a) Let f be a nonnegative function which is Lebesgue integrable over a measurable set E. Given $\epsilon > 0$, show there exists $\delta > 0$ such that for every measurable set $A \subset E$, with $\lambda(A) < \epsilon$, it is true that

$$\int_A f d\lambda < \epsilon.$$

   (Hint: this result is very easy if f is bounded. If not, let $f_n(x) = f(x)$ if $f(x) \leq n$ and let it be n otherwise. Indicate by name any important convergence theorem you use.)

(b) Suppose f is a Lebesgue integrable function on an interval $[a, b]$, and let $F(x) = \int_a^x f d\lambda$. Among the properties *continuity, bounded variation, absolute continuity,* which are held by F? Prove one of them.

5. (a) Is it true that if $f'(x) = 0$ a.e. on an interval, that it must by constant on that interval? If this is not true, do you know a counterexample?

   (b) If you are given a function F which is absolutely continuous on an interval, what can you say about F?

   (c) Suppose that F is a bounded linear functional on $L^p[0,1]$. Given $s \in [0,1]$, let χ_s denote the characteristic function of the interval $[0, s]$. Given that the function $\phi(s) = F(\chi_s)$ is absolutely continuous on $[0,1]$, prove that there is an integrable function g such that

$$F(\psi) = \int_0^1 g\psi d\lambda$$

   for all step functions ψ on $[0,1]$. Do you think that one could use this to show that $F(f) = \int fg$ for all $f \in L^p$?

6. Let $f(x) = x^{-2/3}$ for $x \in [0,1]$.

   (a) Is the function f essentially bounded on $[0,1]$? That is, does $f \in L^\infty[0,1]$?

   (b) Show that f is in $L^1[0,1]$. Give the details and identify any convergence theorems you are using.

   (c) Does $f \in L^p[0,1]$ for any p with $1 < p < \infty$? Explain.

Part 2 Do (only) *five* of the six problems.

1. (a) Determine the locus of the set $\left\{ z \in \mathbb{C} : \mathrm{Im}\left(\frac{z - 1 + i}{2 + i}\right) \leq 0 \right\}$.

   (b) Use simple examples to explain the difference between "f is differentiable at the point a" and "f is analytic at the point a".

2. Evaluate the integrals.

(a) $\displaystyle\int_0^\infty \frac{dx}{4+x^2}$.

(b) $\displaystyle\int_\gamma \frac{e^z\,dz}{(z-2)(z+1)(z-4)}$ for $\gamma(t) = -2+5e^{it}$, $0 \le t \le 2\pi$.

3. Show that the polynomial $z^6+z^4-z^3-6z+1$ has no zeros on the circles $|z| = 1$ and $|z| = 2$. Determine the number of zeros in the annulus $ann(0;1,2) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

4. (a) Give an example of an analytic function that maps the open unit disk onto an interval on the real line.

   (b) Let f be an analytic function on $\mathbb{C}$ such that Re $f(z) > 0$ for all $z \in \mathbb{C}$. What can be said about the function $g(z) = e^{-f(z)}$? What can be said about f?

5. Expand the function $f(z) = \dfrac{3}{z^2-z-2}$ in power series or Laurent series in the region (a) $|z| < 1$; (b) $1 < |z| < 2$; and (c) $1 < |z-1| < 2$.

6. (a) What is an isolated singularity? How many kinds of singularities are there?

   (b) Give an example of an analytic function on the punctured disk $0 < |z| < 1$ such that $|f(z)| \le 2$ for $0 < |z| < 1$. Is the singularity removable? Explain.

   (c) If g has an essential singularity at 0, then what can be said about the behavior of g near 0? Do we have $\lim_{z\to 0} g(z) = \infty$? Why?

Choose only one of Section B and Section C

Section B. Do (only) *two* of the three problems.

1. You may assume standard results about Lebesgue measure for this problem. Let $C \subseteq [0,\ 1]$ be the Cantor middle thirds set.
(a) Prove that C has Lebesgue measure zero.
(b) Give a reason why $C \cap \mathbb{Q} \neq \emptyset$. (A proof is not required.)
(c) For each $y \in \mathbb{R}$, let $C+y = \{x+y:\ x \in C\}$. Prove that $\{y \in \mathbb{R}:\ (C+y) \cap \mathbb{Q} \neq \emptyset\}$ has Lebesgue measure zero. **Hint:** Consider the sets $r - C$ for $r \in \mathbb{Q}$.

2. **(a)** Let X be a set, $E \subseteq X$, and $\mathcal{B}$ be a $\sigma-$algebra in X. Prove that $\{E \cap B:\ B \in \mathcal{B}\}$ is a $\sigma-$algebra in E.
(b) Let $(X,\ \mathcal{A},\ \mu)$ be a $\sigma-$finite measure space. Prove that

$$\{x \in X:\ \{x\} \in \mathcal{A} \text{ and } \mu(\{x\}) \neq 0\}$$

is countable. **Hint:** First prove the result for a finite measure space.

3. **(a)** State the Radon–Nikodym Theorem.
(b) State the Lebesgue Decomposition Theorem for measures.
(c) Do only one of (i) or (ii): **(i)** Assuming that the Radon–Nikodym Theorem holds for finite measures, prove the Radon–Nikodym Theorem holds for $\sigma-$finite measures. **(ii)** Let λ be Lebesgue measure on the Borel subsets of $[0,\ 1]$ and let μ be the counting measure on the Borel subsets of $[0,\ 1]$. These two measures can be used to show that the conclusion of the Radon–Nikodym Theorem can fail without a certain hypothesis. Give a detailed explanation of this.

Section C Do (only) *two* of the three problems.

1. (a) Give an example of a sequence $\{z_n\}$ such that $\prod_{n=1}^{\infty} z_n$ converges but not absolutely.

   (b) Give an example of an entire function with simple zeros at the positive integers. Explain your answer.

2. (a) What does Runge's theorem say?

   (b) Explain why some analytic functions on a certain regions G cannot be approximated in the topology of $H(G)$ by polynomials with an example.

3. (a) Given a continuous real valued function u on a region G, what do we mean by u has the mean value property?

   (b) Prove that every harmonic function has the mean value property.

   (c) Let G be a region; and let $u : G \to \mathbb{R}$ be a continuous function with the mean value property. Must u be harmonic on G? Explain.