Ph. D. Qualifying Examination in Analysis

August 24, 2007

Part 1. Do (only) ***four*** **of the five problems.**

1. (a) Define a *measurable* set.

   (b) Show that if E is a measurable set, then each translate $E+y$ of E is also measurable.

2. (a) Let f be a nonnegative measurable function. Show that $\int f = 0$ implies $f = 0$ a.e.

   (b) Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converge to f, and suppose $f_n \leq f$ for each n. Prove
   $$\int f = \lim \int f_n ,$$
   and identify any theorems you might use.

3. (a) Is the function $f(t) = t^{-2/3}$ integrable in the Riemann sense over the interval $[0,1]$? Give a reason for your answer.

   (b) Is the function $f(t) = t^{-2/3}$ Lebesgue integrable on the closed interval $[0,1]$? Prove your answer, indicating any convergence theorem you are using.

   (c) Is the function $g(t) = t^{1/3}$ absolutely continuous on the closed interval $[0,1]$? Justify your answer.

4. (a) Suppose f is integrable on $[a,b]$. Show that the function $F(x) = \int_a^x f(t)\,dt$ is a continuous function of bounded variation on $[a,b]$.

   (b) What is known about the differentiability of F? Explain.

   (c) Suppose f is bounded and measurable on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt + F(a)$. For
   $$f_n(x) = \frac{F(x+1/n) - F(x)}{1/n}$$
   show
   $$\int_a^c F'(x)\,dx = \lim_{n\to\infty} \int_a^c f_n(x)\,dx = \int_a^c f(x)\,dx$$
   for all c in the closed interval $[a,b]$.

5. (a) Indicate the definition and basic properties of $L^p[0,1]$.

   (b) Suppose F is a bounded linear functional on L^p. What does that mean?

   (c) Given a bounded measurable function f on the closed interval $[0,1]$ is it true or false that there exists a bounded sequence of step functions which converge almost everywhere to f? Prove your answer.

   (d) Suppose that there is an integrable function g on $[0,1]$ such that $F(\psi) = \int_0^1 g\psi$ for each step function ψ on $[0,1]$. Prove that
   $$F(f) = \int_0^1 gf$$
   for each bounded measurable function f on $[0,1]$.

Part 2. Do (only) *four* of the five problems.

1. Let $f(z) = 2z^6 + z^4 - 2z^3 + 8z^2 - e^z$.

   (a) How many zeros does $f(z)$ have in the region $|z| < 1$? Justify your answer.

   (b) How many zeros does $f(z)$ have in the region $1 \leq |z| < |2|$? Justify your answer.

2. Suppose f is an entire function with $f(0) = 1$ and $|f(z)| \leq Ae^x$ for all $z = x + iy \in \mathbb{C}$ and some positive constant A. *Prove*: $f(z) = e^z$ for all z.

3. Expand the function $\dfrac{2z-7}{z^2 - 7z + 10}$ in a Laurent series in the region

   (a) $2 < |z| < 5$.

   (b) $|z| > 5$.

4. Let γ be the rectangle with vertices $-R, R, R + i\pi, -R + i\pi$. Consider the integral of $e^{az}/\cosh z$ around γ. *Prove*: If $|a| < 1$, then

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x} dx = \frac{\pi}{\cos(\pi a/2)}.$$

5. It is desired to approximate $1/z$ on the circle $|z| = 1$ by a polynomial $p(z)$. Prove that the maximum error is at least 1; that is,

$$\max_{|z|=1} \left| \frac{1}{z} - p(z) \right| \geq 1.$$

   Hint: Integrate $1/z - p(z)$ around the circle $|z| = 1$.

Part 3. Do (only) *two* of the three problems.

1. (a) Let $\mathcal{F}$ be a family of real-valued functions defined on a Banach space X and suppose that for each $x \in X$, there is a real number M_x such that $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$.

   i. For each positive integer m and $f \in \mathcal{F}$, let

$$E_{m,f} = \{x : |f(x)| \leq m\} \text{ and } E_m = \bigcap_{f \in \mathcal{F}} E_{m,f}.$$

   Argue that each E_m is closed.

   ii. Show that $X = \bigcup_{m \in \mathbb{N}} E_m$.

   iii. Argue that there must exist some open set G in X and some positive integer m such that $|f(x)| \leq m$ for all $f \in \mathcal{F}$. (You may state a pertinent theorem.)

   (b) Suppose $\mathcal{F}$ is a family of bounded linear operators from a Banach space X to a normed linear space Y such that for each $x \in X$, there exists $M_x > 0$ such that $||Tx|| \leq M_x$ for all $T \in \mathcal{F}$.

   i. Argue that there exists an open set G and a constant K such that $||Tx|| \leq K$ for all $x \in G$.

ii. Show that there is a constant M so that $||T|| \leq M$ for all $T \in \mathcal{F}$.

(c) What theorem is being proved above?

(d) If $\{f_n\}$ is a sequence of linear functionals on a Banach space X such that $\{f_n(x)\}$ is bounded for each $x \in X$, can you conclude that $\{||f_n||\}$ is bounded?

(e) Answer the same question for a sequence $\{x_n\}$ in a Banach space X for which $\{f(x_n)\}$ is bounded for each $f \in X^*$.

2. Let $X = \{a,b,c\}$, $\Sigma = \{\emptyset, X, \{a,b\}, \{c\}\}$, and define μ by $\mu(X) = 1, \mu(\emptyset) = \mu(\{a,b\}) = 0, \mu(\{c\}) = 1$. Let f be identically 1 on X, and let g be defined by $g(a) = 0, g(b) = g(c) = 1$.

   (a) Does (X, Σ, μ) define a measure space?

   (b) Is the function f measurable?

   (c) Does $f = g$ a.e.?

   (d) Is g measurable?

   (e) Is the following statement true in general: If f is measurable on (X, Σ, μ) and $f = g$ a.e., then g is measurable?

   (f) If the statement in the previous part is true, prove it. If it is false, make it into a true statement and prove it.

3. (a) Define each of the following.

   i. An outer measure on a set X.

   ii. A measurable set with respect to an outer measure.

   iii. An algebra of sets and a σ-algebra of sets.

   iv. A measure on an algebra.

   v. The measure induced by an outer measure.

   vi. A semi-algebra of sets.

   (b) Given complete measure spaces $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$, explain how to define the product measure $\mu \times \nu$ on the product space $X \times Y$.

   (c) Prove the following statement which is needed in the development of part (b).
   Let $\{(A_j \times B_j)\}$ be a countable disjoint collection of measurable rectangles whose union is a measurable rectangle $A \times B$. Then

$$\mu(A)\nu(B) = \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j).$$