Analysis Qualifying Exam

August 29, 2008

Part 1: MTH 632: Introduction to Real Analysis

Provided complete solutions of 4 of the 5.

1. Let $E \subset \mathbb{R}$ be Lebesgue measurable. Show $E = G \cup Z$ where G is a $\mathcal{G}_\delta$ set and Z is a set of measure zero. Does the converse hold, i.e., if a set $E = G \cup Z$, G a $\mathcal{G}_\delta$, and $m(Z) = 0$, is E measurable?

2. Let f be a a Lebesgue integrable function on $\mathbb{R}$. Let $\langle E_n \rangle$ be a sequence of measurable sets such that $\lim_{n\to\infty} m(E_n) = 0$. Show that $\lim_{n\to\infty} \int_{E_n} f = 0$.

3. Let f be a Lebesgue measurable function defined on $[0,1]$. Suppose $0 \le f(x) \le 1$ and $\int_{[0,1]} f\,dm = 1$. Show that $f = 1$ a.e.

4. Let $\langle f_n \rangle$ be a sequence of measurable functions defined of $[0,1]$ and f a measurable function. Assume that, for any $\epsilon > 0$ there exists measurable $E_\epsilon \subset [0,1]$ such that $m([0,1] \sim E_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on E_ϵ. Show that $f_n \to f$ a.e.

5. For f a measurable function on $[0,1]$ let $E_n = \{x \in [0,1] : n-1 \le |f(x)| < n\}$.

   (a) Show $f \in L^1[0,1]$ if and only if $\sum_{n=1}^{\infty} n\, m(E_n) < \infty$.

   (b) What can be said about $f \in L^p[0,1]$, $1 \le p < \infty$ and the sets $\langle E_n \rangle$? Prove your answer.

Part 2: MTH 636: Introduction to Complex Variables

Provide complete solutions to 5 of the 6 problems.

1. (a) Find the set of points at which the function $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable. What is the set of points at which h is analytic? Justify your answers.

   (b) Give an example of a non constant analytic function $w = f(z)$ that maps the half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Re } z > 1\}$ onto the interval $(0, 1)$ in the x-axis. Prove your answer.

2. Let f be analytic on an open set $\mathcal{O} \subsetneq \mathbb{C}$; and let $a \in \mathcal{O}$. Suppose there are an $r > 0$ and a sequence $\{a_k\}$ such that $\{z \in \mathbb{C} : |a - z| < r\} \subseteq \mathcal{O}$ and $f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$ converges for all z satisfying $|z - a| < r$. Let $z \in \mathbb{C}$ satisfy $|z - a| < \delta := \text{dist}(z, \mathbb{C}\setminus\mathcal{O}) := \inf_{w \in \mathbb{C}\setminus\mathcal{O}} |a - w|$. Choose η such that $|z - a| < \eta < \delta$.

   (a) Express a_k in terms of f and the circle $|\zeta - a| = \eta$.

   (b) Show that the series $\sum_{k=0}^{\infty} \frac{(z-a)^k}{(\zeta - a)^{k+1}} f(\zeta)$ converges uniformly and absolutely on the circle $|\zeta - a| = \eta$.

   (c) Use parts (a) and (b) to prove that the series $\sum_{k=0}^{\infty} a_k (z-a)^k$ converges for all z satisfying $|z - a| < \delta$ and its sum is $f(z)$.

3. (a) If $\sum_{k=0}^{\infty} a_k z^k$ converges for some $z_0 \neq 0$ and if $r \in (0, |z_0|)$, then $\sum_{k=0}^{\infty} a_k z^k$ converges uniformly and absolutely for all z satisfying $|z| \leq r$.

   (b) Suppose f is an analytic function on $\{z \in \mathbb{C} : |z| > 1\}$ and $\lim_{z \to \infty} f(z) = 0$. If γ is the positively oriented circle of radius 2 centered at 0 and $|z_0| > 2$, find $\int_\gamma \frac{f(z)}{z - z_0} \, dz$. (*Hint:* Join *large* circles centered at 0 of radius $R > |z_0|$ with the circle of radius 2 by radial segments.)

4. If f is analytic on $\mathbb{C} \setminus \{0\}$ and if $|f(z)| \leq 1$ for all $z \neq 0$, what can be said about f? Prove your answer.

5. (a) Compute $\displaystyle\int_{|z-1|=2} \left(z \left[\exp\left(\frac{2}{z^2}\right) - 1 \right] + \frac{\sin^2 z}{z^2 (2z - \pi)^2} \right) dz$ (where $|z - 1| = 2$ is the positively oriented circle of radius 2 centered at 1).

   (b) Evaluate $\displaystyle\int_0^\infty \frac{x^2 \, dx}{(x^2 + 2x + 2)^2}$.

6. (a) Find the number of solutions of $2z^5 - 7z^2 - z = 1 - z^4$ in the annulus

$$\{z \in \mathbb{C} : 1 < |z| < 2\}.$$

(b) If f is an entire function that satisfies $|f(z)| \leq 100|z|^2$ for all z satisfying $|z| > 100$, then f must have a simple form. What is it? Prove your answer.

Part 3: MTH 637: Complex Variable Theory

1. (a) Give an example of a planar region G and a harmonic function $u : G \to \mathbb{R}$ such that u does not have a harmonic conjugate.

   (b) Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ is a non-constant harmonic function, then u is onto.

2. Show that there is an analytic function on the open unit disk $\mathbb{D}$ which is not analytic on any region containing $\mathbb{D}$.

3. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a non-polynomial analytic function on a region G, and let S_n be the nth partial sum. Show that for every n there exists $z_n \in G$ such that $|S_n(z_n)| > |f(z_n)|$.

4. Let f be an entire function such that the equations $f(z) = 0$ and $f(z) = 1$ each have finitely many solutions. Show that f is a polynomial.