Analysis Qualifying Exam: August, 2010

MTH 632: Provide complete solutions to 5 of the 6 questions.

Notation: $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{R}$ denotes the set of real numbers.

1. Let $E \subset \mathbb{R}$ be such that $m(E) < \infty$. Suppose f and the sequence $\langle f_n \rangle$ be measurable functions on E such that $f_n \to f$ a.e. Show there exists measurable sets $G, E_k \subset E$ such that $E = G \cup (\cup_{k=1}^{\infty} E_k)$, $m(G) = 0$, and $f_n \to f$ uniformly on each H_k.

2. Let $\mathcal{M}$ be the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$ and f be a function defined on $\mathbb{R}$ with values in the set of extended real numbers.
   (a) Show that if f is measurable then $f^{-1}(\{\infty\}) \in \mathcal{M}$ and $f^{-1}(O) \in \mathcal{M}$ for all open subsets $O \subset \mathbb{R}$.
   (b) Show that if $f^{-1}(\{\infty\}) \in \mathcal{M}$ and $f^{-1}(B) \in \mathcal{M}$ for all Borel sets $B \subseteq \mathbb{R}$, then f is measurable.

3. Let $f \in L^1[0, \infty)$ and define

$$g(y) = \int_0^{\infty} f(x) \sin(xy)\, dx.$$

   Show that g is a bounded and continuous function of y on all of $\mathbb{R}$.

4. Let f be a real-valued integrable function defined on $\mathbb{R}$. Let $\langle E_n \rangle$ be a sequence of measurable sets such that $\lim_{n\to\infty} m(E_n) = 0$. Show that $\lim_{n\to\infty} \int_{E_n} f\, dm = 0$.

5. Suppose f and g are absolutely continuous functions defined of the interval $[a, b]$. Show fg, their product, is also absolutely continuous.

6. Let $E \subset \mathbb{R}$ have finite Lebesgue measure.
   (a) Let $1 \le p < q \le \infty$. Show $L^q(E) \subset L^p(E)$ and $\| f \|_p \le \| f \|_q \, (m(E))^{1/p - 1/q}$ for any $f \in L^p(E)$.
   (b) If $1 \le p < q < r \le \infty$. Show $L^q(E) \subset L^p(E) + L^r(E)$. Explain why this show that each $f \in L^q(E)$ can be written as a sum of two functions, $f = g + h$, $g \in L^p(E)$, $h \in L^r(E)$. (Hint: For $f \in L^q(E)$, consider the two sets $G = \{x \in E : |f(x)| > 1\}$ and $H = \{x \in E : |f(x)| \le 1\}$.)