Analysis Qualifying Exam August 19, 2014

MTH 636: Provide complete solutions to **only** five of the six problems.

1. Each of the following functions has a singularity at $z = 0$. Determine whether the singularity is an essential singularity, a pole, or a removable singularity. If the singularity is a pole, determine the order of the pole. Justify your answers.

   (a) $\dfrac{z - \sin z}{z^3}$

   (b) $\dfrac{(z + z^{-1})^2}{z}$

2. Find the Laurent expansion of the function

$$f(z) = \frac{2z + 1}{z^2 + z - 6}$$

   (a) in the annulus $2 < |z| < 3$, and

   (b) in the region $|z| > 3$.

3. (a) Prove that all of the roots of $z^5 - z^4 + z^2 - z + 6$ lie in the annulus $1 \le |z| \le 2$.

   (b) Suppose a polynomial $p(z)$ has a zero at z_0. What is the residue of $\dfrac{p'}{p}(z)$ at z_0?

   (c) Compute

$$\frac{1}{2\pi i} \oint_{|z|=2} \frac{5z^4 - 4z^3 + 2z - 1}{z^5 - z^4 + z^2 - z + 6} dz.$$

4. Suppose that f is an entire function with the property there is some real number α such that if $z = x + iy$ is an arbitrary complex number, then $|f(z)| \le e^{\alpha x}$. Prove that $f(z) = \lambda e^{\alpha z}$ for some $\lambda \in \mathbb{C}$.

5. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2 + \pi^2} dx.$$

6. Suppose that a is a real number with $|a| > 1$. **Prove**:

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi}{a^2 - 1}.$$

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

1. Let D be a bounded set in $\mathbb{R}$, with the property that for any interval I,

$$m^*(D \cap I) \leq \frac{1}{2}\text{length}(I),$$

where m^* denotes the exterior or outer Lebesgue measure. Show that D is measurable and has measure zero.

2. (a) State Fatou's Lemma and the Monotone Convergence Theorem.

   (b) Use Fatou's Lemma to prove the Monotone Covergence Theorem.

3. Let f be a measurable function on a bounded interval J, which is finite a.e. on J. Given any $\epsilon > 0$, show that there exists a constant M such that $|f(x)| < M$ for all $x \in J$ outside of some set of measure $< \epsilon$.

4. Let $f_k \to f$ a.e. on $\mathbb{R}$. Show that given $\epsilon > 0$, there exists a set E of measure less than ϵ, so that $f_k \to f$ uniformly on $I \setminus E$, for any finite interval I.

5. Compute the following limit and justify your calculation:

$$\lim_{n\to\infty} \int_0^\infty \frac{x^{n-2}}{1+x^n} \cos(\pi n x)\, dx$$

6. Let $1 \leq r < p < s < \infty$.

   (a) Show that $L^r(\mathbb{R}) \cap L^s(\mathbb{R}) \subset L^p(\mathbb{R})$.

   (b) Show that $L^s([0,1]) \subset L^r([0,1])$.

   (c) Is it true that $L^s(\mathbb{R}) \subset L^r(\mathbb{R})$? Justify your answer.