Central Michigan University
Department of Mathematics
Analysis Qualifying Examination
August 26, 2015

INSTRUCTIONS

(1) This question paper has 6 problems from Real Analysis (MTH 632) numbered **R1** through **R6** and 6 problems from Complex Analysis numbered **C1** through **C6**.
(2) Do **five** problems from the Real Analysis (MTH 632) part, and another **five** problems from the Complex Analysis (MTH 636) part.
(3) Begin each problem on a separate sheet of paper, and clearly write the problem number and your name.
(4) At the end, submit the Real Analysis and Complex Analysis solutions in **separate** bunches.

Good Luck!

MTH 636: Do **five** of the following six :

(C1) (a) Find the Laurent expansion of the function $f(z) = \dfrac{1}{z-2}$ in the annulus

$$\left\{ z \in \mathbb{C}\colon \sqrt{5} < |z - i| < \infty \right\}.$$

(b) Find the number of roots (counting multiplicity) of the equation

$$3z^{2015} + z^2 + 1 = 0$$

which lie in the unit disc $\{z \in \mathbb{C}\colon\ |z| < 1\}$.

(C2) Locate all isolated singularities of the function f on $\mathbb{C}$ given below, and classify these singularities as removable singularities, poles and essential singularities. For each pole, find the order.

$$f(z) = \frac{(z-1)(z-2)^2 \cdot \sin\left(\frac{1}{z}\right)}{\sin^2(\pi z)}.$$

(C3) (a) Suppose that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is equal to 1. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} b_n z^n$, where

$$b_n = \left(1 + \frac{1}{n}\right)^{n^2} \cdot a_n.$$

(b) In each of (i) and (ii) below, find all holomorphic functions f on the unit disc $\{z \in \mathbb{C}\colon\ |z| < 1\}$ satisfying the given conditions, or show that no such function f exists. Give complete justification for all your claims.

(i) $f\left(\frac{1}{n}\right) = 0$, for each positive integer $n \geq 2$.

(ii) $|f(z)| = 1 + |z|^2$ for each z in the unit disc. (Hint: Consider the function $\frac{1}{f}$.)

(C4) Let f be a holomorphic function on the unit disc such that if we write $f = u + iv$ with u, v real valued, then $v(z) = u(z)^2$ for each z in the unit disc. If $u(0) = 1$, find the function f.

(C5) Let f be a holomorphic function on the upper half-plane $H = \{z \in \mathbb{C}\colon\ \mathrm{Im}(z) > 0\}$, such that for each $z \in H$ we have $|f(z)| < 1$. Show that the n-th derivative of f satisfies

$$\left|f^{(n)}(z)\right| \leq \frac{n!}{(\mathrm{Im}\, z)^n}.$$

(C6) (a) Let Γ be the unit circle $\{z \in \mathbb{C}\colon\ |z| = 1\}$ oriented counterclockwise. Compute the line integral

$$\int_\Gamma \frac{\sin(\pi z)}{\left(z - \frac{1}{2}\right)(z-2)} dz.$$

(b) Use the method of residues to compute the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2}.$$

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) Let E be a set with finite outer measure. Prove that if E is not measurable, then there is an open set U containing E with $m^*(U) < \infty$ and
$$m^*(U \setminus E) > m^*(U) - m^*(E).$$

(R2) Let E be measurable, $1 \leq p < \infty$ and q the conjugate of p. Prove that if $g \in L^p(E)$ has the property that $\int_E f \cdot g = 0$ for every $f \in L^q(E)$ then $g = 0$ a.e.

(R3) (a) State Fatou's Lemma.

(b) Suppose that $\{f_n\}$ is a sequence of non-negative measurable functions on $\mathbb{R}$ that converge pointwise to f on $\mathbb{R}$, and
$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f.$$
Prove that for every measurable set E,
$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

(R4) Let $\{f_n\}$ be a sequence of functions in $L^1(\mathbb{R})$, with $\|f_n\|_{L^1} \leq 1$ for all n. Define
$$f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$
Prove that f is Lebesgue measurable and that $\|f\|_{L^1} \leq 1$.

(R5) (a) Give the definition of an absolutely continuous function.

(b) Let f be absolutely continuous on $\mathbb{R}$ with
$$\lim_{t\to 0^+} \int_{\mathbb{R}} \frac{|f(x+t) - f(x)|}{t} = 0.$$
Prove that f is constant.

(R6) Let $\{f_n\}$ be a sequence of measurable functions on E that converges to the real-valued function f pointwise on E. Prove that $E = \bigcup_{k=1}^{\infty} E_k$, where for each $k \geq 1$, E_k is measurable, $\{f_n\}$ converges uniformly to f on E_k for all $k \geq 2$, and $m(E_1) = 0$.