Central Michigan University

Department of Mathematics

Analysis Qualifying Examination

August 24, 2016

INSTRUCTIONS

(1) This question paper has 6 problems from Complex Analysis (MTH 636) numbered **C1** through **C6** and 6 problems from Real Analysis (MTH 632) numbered **R1** through **R6**.
(2) Do **five** problems from the Real Analysis (MTH 632) part, and another **five** problems from the Complex Analysis (MTH 636) part.
(3) In either section, if you attempt solutions to all six problems, then only the five highest scores will count.
(4) Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.
(5) Give proper mathematical justification of all your statements.
(6) At the end, turn in the Real Analysis and Complex Analysis solutions in **separately stapled** bunches.

Good Luck!

MTH 636: Do <u>**five**</u> of the following six :

(C1) Determine where the function $f(z) = \overline{z}\mathrm{Re}\, z + z\mathrm{Im}\, z + \overline{z}$ has a complex derivative and compute the derivative wherever it exists.

(C2) Let $U \subset \mathbb{C}$ be and open connect set that contains the disk $D = \{z : |z - z_0| \leq R\}$. Let $f \in H(U)$ be such that $|f(z)| \leq M$ for all $z \in \partial D = \{z : |z - z_0| = R\}$. Let $z_1, z_2 \in \left\{z : |z - z_0| \leq \frac{1}{2}R\right\}$. Show

$$|f(z_1) - f(z_2)| \leq \frac{4M}{R}\, |z_1 - z_2|.$$

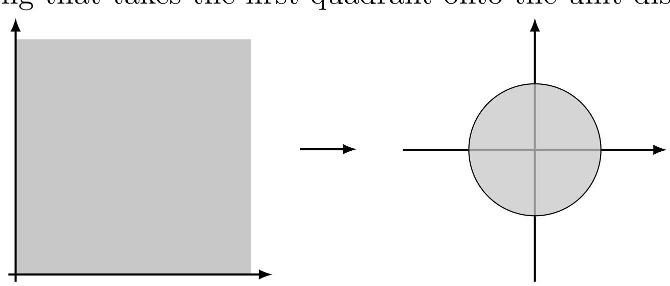
(C3) Let f be an entire function such that $f(z + m + ni) = f(z),\ z \in \mathbb{C},\ m, n \in \mathbb{Z}$. Prove that f is constant.

(C4) Let $U = \{z \in \mathbb{C} : |z| > 1\}$ and suppose $f \in H(U)$. Suppose $\lim_{z\to\infty} f(z) = 0$. Show that if $|z| > 2$, then

$$\frac{1}{2\pi i}\int_{|w|=2} \frac{f(w)}{w - z}\, dw = -f(z).$$

(C5) Evaluate the integral $\displaystyle\int_0^\infty \frac{\cos x}{(1 + x^2)^2}\, dx$.
(Hint: Consider the contour $\gamma_R = \{Re^{it} : 0 \leq t \leq \pi\} \cup [-R, R]$, $R > 1$.)

(C6) Use a linear-fractional transformation in composition with another map to find a conformal mapping that takes the first quadrant onto the unit disk.



MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at each point of $\mathbb{R}$. Show that the derivative $f' : \mathbb{R} \to \mathbb{R}$ is a measurable function.
(b) Denote by E the set of real numbers x such that $0 < x < 1$, and x has a decimal representation in which each digit is a 2 or a 4. Find the outer measure of E and show that E measurable.

(R2) Let $\{f_n\}$ be a sequence of continuous functions on $[0,1]$, such that each $f_n \geq 0$, and $\int_{[0,1]} f_n = 1$. Suppose f is a function on $[0,1]$ such that $f_n \to f$ pointwise as $n \to \infty$.
State which of the following statements are **necessarily true** and which **may be false**. For each true statement, give a justification why it is true. For each false statement, give a counterexample.
(a) f is continuous.
(b) $\int_{[0,1]} f \leq 1$.
(c) $\int_{[0,1]} f = 1$.
(d) Suppose that for each integer $n \geq 1$ and for each $x \in (0,1)$, we have

$$f_n(x) \leq 4x^{-\frac{2}{3}}.$$

Then $\int_{[0,1]} f = 1$.
(e) For each n, we have $\int_{[0,1]} f_n^2 \geq 1$.

(R3) Let $f \in L^1([0,1])$, and set for $x \in \mathbb{R}$:

$$g(x) = \int_{[0,1]} f(y) \sin^2(xe^y)\, dy,$$

where the integral is a Lebesgue integral. Show that $g : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function.

(R4) Let $E \subset \mathbb{R}$ be a measurable set such that $0 < m(E) < \infty$. For a function $f \in L^p(E)$, set

$$N_p(f) = \left(\frac{1}{m(E)} \int_E |f|^p \right)^{\frac{1}{p}}.$$

Let $1 \leq p_1 < p_2 < \infty$ and suppose that $f \in L^{p_2}(E)$. Show that

$$N_{p_1}(f) \leq N_{p_2}(f).$$

(R5) (a) Give an example of a function $f : [0,1] \to \mathbb{R}$ which is differentiable almost everywhere, and such that the function $f' \in L^1([0,1])$, but we have

$$\int_{[0,1]} f' < f(1) - f(0).$$

(b) Suppose that f and g are functions of bounded variation on the interval $[0,1]$. Show that the product fg is also a function of bounded variation.

(R6) For (a) through (d), either give an example of the object described, or state that such an object does not exist. If such an object does not exist, justify your answer.

(a) A measurable set $M \subset \mathbb{R}$ and a nonmeasurable set $N \subset \mathbb{R}$ such that $M \cap N = \emptyset$ and $M \cup N$ is measurable.

(b) A nonmeasurable function $f : \mathbb{R} \to \mathbb{R}$ such that f^2 is measurable.

(c) A function which belongs to $L^3(\mathbb{R})$ but not to $L^2(\mathbb{R})$.

(d) A sequence of measurable functions $\{f_n\}$ on $[0,1]$ such that as $f_n \to f$ pointwise on $[0,1]$, but

$$\lim_{n\to\infty} \int_{[0,1]} \frac{1}{1+f_n^2}$$

does not exist.