Analysis Qualifying Exam January 10, 1997, 1:00-5:00

Section A Answer 10 of the 12.

1. Suppose f has derivatives of all orders on [-a,a] and $|f^n(x)| \leq |n \sin nx|$ for all n and all $x \in$ [-a,a].

   a) If you wanted to represent f by a series of some kind, what type of series would you use? Explain your answer.

   b) Can you show that the series chosen in part (a) represents f on [-a,a]? If so, do so.

   c) Is the convergence of the above series uniform? How do you know?

2. Consider the sequence $\{u_n(x)\}$ defined by $u_1(x) = \frac{x}{e^x}$,
$u_2(x) = \frac{x}{e^{2x}} - \frac{x}{e^x}$, $u_3(x) = \frac{x}{e^{3x}} - \frac{x}{e^{2x}}$, ..., $u_n(x) = \frac{x}{e^{nx}} - \frac{x}{e^{(n-1)x}}$, ...

   a) Show that the series $\sum_{n=1}^{\infty} u_n(x)$ converges pointwise to 0 on $[0, +\infty)$.

   b) Does the series $\sum_1^{\infty} u_n(x)$ converge uniformly on $[0, +\infty)$? Justify your answer. (Hint: Does $\frac{x}{e^{nx}}$ have a max on $[0, +\infty)$?)

   c) Is each $u_n(x)$ R-integrable on [0,b] for every b> 0? How do you know?

   d) Does the series $\sum_1^{\infty}\left(\int_0^b u_n(x)dx\right)$ converge? If so, to what does it converge?

   e) Can you say anything about the series $\sum_1^{\infty} u_n'(x)$?

3. a) Show that a sequence $\{\mathbf{x}_m\}$ in $\mathbf{R}^n$ converges to $\mathbf{x}$ in $\mathbf{R}^n$, if and only if the sequence $\{\mathbf{x}_m\}$ converges to $\mathbf{x}$ coordinatewise.

   b) Show that a function $\mathbf{F} = (f_1, f_2, \ldots f_n)$ from $\mathbf{R}^m$ to $\mathbf{R}^n$ is continuous at $\mathbf{x}_0$ in $\mathbf{R}^m$ if and only if each f_j is contiuous at $\mathbf{x}_0$.

   c) Show that if $\{\mathbf{x}_m\}$ is a bounded sequence in $\mathbf{R}^n$, then it must have a convergent subsequence in $\mathbf{R}^n$.

4. Let $\mathbf{F} = (f_1, f_2, f_3)$: $\mathbf{R}^5 \to \mathbf{R}^3$ be given by
$$f_1(x_1, x_2, x_3, x_4, x_5) = x_1 + 5x_2 + x_3 - x_4 + x_5$$
$$f_2(x_1, x_2, x_3, x_4, x_5) = x_3 + 2x_4 - x_5$$
$$f_3(x_1, x_2, x_3, x_4, x_5) = x_1 + 7x_2 + 2x_3 - 3x_4 + 2x_5$$

   a) What does it mean to say that a function such as $\mathbf{F}$ above is differentiable at a given point?

   b) Is the function $\mathbf{F}$ above differentiable at $\mathbf{0} = (0, 0, 0, 0, 0)$? How do you know? (You may quote pertinent theorems.) Is $\mathbf{F}$ differentiable at every point of $\mathbf{R}^5$?

c) If the answer in part (b) is yes, compute the derivative of **F** at **0**.

d) Can the system $\mathbf{F}(x_1, x_2, x_3, x_4, x_5) = \mathbf{0}$ be solved for x_4, x_5 in terms of x_1, x_2, x_3? Are there differentiable functions g_1, g_2 on an open set in $\mathbf{R}^3$ so that $x_4 = g_1(x_1, x_2, x_3)$; $x_5 = g_2(x_1, x_2, x_3)$? What theorem guarantees answers to these questions and how does it apply?

5. a) Give the development of Lebesgue measure and the Lebesgue integral, starting with the definition of outer measure and concluding with the important convergence theorems. State carefully the pertinent definitions and theorems.

b) Give an example to show the use of one of the important convergence theorems to establish the Lebesgue integrability of a function.

6. Prove that $\int_0^\infty e^{-x}\sin(\sqrt{x})dx = \sqrt{\pi}\sum_1^\infty \frac{(-1)^n(1\cdot3\ldots(2n-1))}{(2n-1)!\,2^n}$. You may use the facts that ?

$$\sin\sqrt{x} = \sum_1^\infty \frac{(-1)^{n+1}x^{\frac{2n-1}{2}}}{(2n-1)!}, \quad e^{x/2} = \sum_0^\infty \frac{x^n}{n!2^n}, \quad \int_0^\infty e^{-x}x^{(n-\frac{1}{2})}dx = \Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot3\ldots(2n-1)^{(2n-1)}}{2^n}\sqrt{\pi},$$

(recall that $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x>0$) and $(2n-1)! \geq 2^{n-1}n!$.

$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n\, n!}$

a) First show that $|e^{-x}\sin\sqrt{x}| \leq 2x^{-1/2}[e^{-x/2} - e^{-x}]$.

b) Observe that the inequality in a) shows that $e^{-x}\sin\sqrt{x}$ is integrable on $[0, \infty)$.

c) Prove the statement given in the beginning.

(See Widder, Advanced Calculus p. 373)

7. a) Show that if f is integrable on [a,b], then f is finite a.e. Does this mean f is bounded on [a,b]?

b) Let $f_n(x) = \begin{cases} k/(k+1) & \text{for } k-1 \leq x < k \text{ and } k = 1,2,\ldots,n. \\ 0 & \text{otherwise} \end{cases}$ Does f_n converge pointwise to a function f on $[0, \infty)$? If so, calculate $\int_0^\infty f$.

c) Does the sequence $\{f_n\}$ in part b) converge in measure?

8. a) What does it mean to say that $f \in L^p[0, 1]$ for some p with $1 \leq p \leq \infty$?

b) Suppose F is a bounded linear functional on L^p. What does that mean?

c) Suppose F is a bounded linear functional on L^p and $\chi_s = \chi_{[0,s]}$ for $s \in [0,1]$. Let $\Phi(s) = F(\chi_s)$. Show that Φ is absolutely continuous. (Hint: For a collection $\{(s_i, s_i')\}_{i=1}^n$ of nonoverlapping intervals, consider the function $f = \Sigma(\chi_i - \chi_i')\text{sgn}(\Phi(s_i) - \Phi(s_i'))$.

d) Based on the result in c), there exists an integrable function g on [0,1] such that $\Phi(s) = \int_0^s g$. How do we know this?

e) From d) we have $F(\chi_s) = \int_0^1 g\cdot\chi_s$, and so $F\psi = \int_0^1 g\psi$ for any step function ψ.

f) Given any bounded measurable function f on [0,1], there is a bounded sequence $\{\psi_n\}$ of step functions which converge almost everywhere to f. Use this fact and part e) to show $F(f)=\int fg$.

9. Let f(z) be an entire function and $M(r) = \max_{|z|=r} |f(z)|$. If $M(r) < 10 \log r$ for all $r > 1998$, prove that f is a constant function.

$\max_{|z|=r}$

10. Let $F(z) = \int_{-1}^{1} \sqrt{|t|}\, \sin(zt)dt$.

a) Prove that F is continuous.
b) State Morera's Theorem.
c) Prove that F is an entire function.

11. Find the Laurent series for $\frac{1}{(z-1)(z+2)}$ in

a) the region $1 < |z| < 2$.
b) the region $|z|$ [illegible] 2. $|z|>2$.

12. a) State Rouche's Theorem.
b) How many roots does the equation $z^4+8z^3-3z-8=0$ have in $|z| < 2$?

Section B Select one of the two parts.

Part 1 Answer 2 of the three.

1. Define the signed measure μ on the σ-algebra M of Lebesque measurable subsets of [0,1] by $\mu(E) = m(E \cap [0, \frac{1}{2}]) - m(E \cap (\frac{1}{2}, 1])$, for all $E \in M$, where m denotes Lebesgue measure.

a) Give two different Hahn decompositions for μ.
b) What is the Jordan decomposition of μ?
c) What is the total variation $|\mu|$?
d) Is μ absolutely continuous with respect to Lebesgue measure?
e) Is there a Radon-Nikodym derivative $\left[\frac{d\mu}{dm}\right]$? Calculate it if it exists.

2. Let (X, A, μ), (Y, B, ν) be two complete measure spaces and $k \in L^2(\mu\times\nu)$. Show that the map T defined by $Tf(x) = \int_Y k(x,y)f(y)d\nu(y)$

a) is a linear transformation from $L^2(\nu)$ to $L^2(\mu)$;
b) satisfies $\|Tf\|_2 \le \|k\|_2\|f\|_2$ for all $f \in L^2(\nu)$

3. Let (X, A, μ) be a measure space such that there exists a sequence $\{X_n\}$ of disjoint measurable subsets of X_n with $\mu(X_n) > 0$ for all n. Show that $L^\infty(X, A, \mu)$ is not separable. Use this to deduce the non-separability of ℓ^∞ and $L^\infty[0, 1]$.

Part 2 Answer 2 of the three.

1. Find a linear fractional transformation which carries $|z| = 1$ and $\left|z - \frac{1}{4}\right| = \frac{1}{4}$ into concentric circles. What is the ratio of their radii?

2. Solve the Dirichlet problem for the disk $D = \{z: |z| < 2\}$ with boundary data $h(z) = x^2 + 2\,xy^2$.

3. Show that $\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)$ is entire if and only if $\sum_{n=1}^{\infty} \frac{1}{z - a_n}$ is meromorphic.